Metoda szeregów potęgowych - przykład P1

Znaleźć rozwiązanie szczególne zagadnienia początkowego

$y'' + x^2 \cdot y' + \ln(1+x) \cdot y = x; \quad y(0) = 0; \quad y'(0) = 1$

w otoczeniu punktu $x = 0$

Współczynniki a_0 oraz a_1 wyznaczamy na podstawie warunków brzegowych.

$a_0 := 0 \qquad a_1 := 1$

$y(x) := a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + a_5 \cdot x^5 + a_6 \cdot x^6 + a_7 \cdot x^7 + a_8 \cdot x^8 + a_9 \cdot x^9$

$y'(x) := \frac{d}{dx} y(x) \rightarrow 9 \cdot a_9 \cdot x^8 + 8 \cdot a_8 \cdot x^7 + 7 \cdot a_7 \cdot x^6 + 6 \cdot a_6 \cdot x^5 + 5 \cdot a_5 \cdot x^4 + 4 \cdot a_4 \cdot x^3 + 3 \cdot a_3 \cdot x^2 + 2 \cdot a_2 \cdot x + 1$

$y''(x) := \frac{d^2}{dx^2} y(x) \rightarrow 72 \cdot a_9 \cdot x^7 + 56 \cdot a_8 \cdot x^6 + 42 \cdot a_7 \cdot x^5 + 30 \cdot a_6 \cdot x^4 + 20 \cdot a_5 \cdot x^3 + 12 \cdot a_4 \cdot x^2 + 6 \cdot a_3 \cdot x + 2 \cdot a_2$

$\log(x) := \ln(1+x) \text{ series}, x, 8 \rightarrow x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8}$

$\log(x) := 1 \cdot x - \frac{1}{2} \cdot x^2 + \frac{1}{3} \cdot x^3 - \frac{1}{4} \cdot x^4 + \frac{1}{5} \cdot x^5 - \frac{1}{6} \cdot x^6 + \frac{1}{7} \cdot x^7$

$F(x) := y''(x) + x^2 \cdot y'(x) + \log(x) \cdot y(x)$

$$W := F(x)\ \text{coeffs}, x, \text{degree} \rightarrow \begin{pmatrix} 2\cdot a_2 & 0 \\ 6\cdot a_3 & 1 \\ 12\cdot a_4 + 2 & 2 \\ 3\cdot a_2 + 20\cdot a_5 - \frac{1}{2} & 3 \\ 4\cdot a_3 - \frac{a_2}{2} + 30\cdot a_6 + \frac{1}{3} & 4 \\ \frac{a_2}{3} - \frac{a_3}{2} + 5\cdot a_4 + 42\cdot a_7 - \frac{1}{4} & 5 \\ \frac{a_3}{3} - \frac{a_2}{4} - \frac{a_4}{2} + 6\cdot a_5 + 56\cdot a_8 + \frac{1}{5} & 6 \\ \frac{a_2}{5} - \frac{a_3}{4} + \frac{a_4}{3} - \frac{a_5}{2} + 7\cdot a_6 + 72\cdot a_9 - \frac{1}{6} & 7 \\ \frac{a_3}{5} - \frac{a_2}{6} - \frac{a_4}{4} + \frac{a_5}{3} - \frac{a_6}{2} + 8\cdot a_7 + \frac{1}{7} & 8 \\ \frac{a_2}{7} - \frac{a_3}{6} + \frac{a_4}{5} - \frac{a_5}{4} + \frac{a_6}{3} - \frac{a_7}{2} + 9\cdot a_8 & 9 \\ \frac{a_3}{7} - \frac{a_4}{6} + \frac{a_5}{5} - \frac{a_6}{4} + \frac{a_7}{3} - \frac{a_8}{2} + 10\cdot a_9 & 10 \\ \frac{a_4}{7} - \frac{a_5}{6} + \frac{a_6}{5} - \frac{a_7}{4} + \frac{a_8}{3} - \frac{a_9}{2} & 11 \\ \frac{a_5}{7} - \frac{a_6}{6} + \frac{a_7}{5} - \frac{a_8}{4} + \frac{a_9}{3} & 12 \\ \frac{a_6}{7} - \frac{a_7}{6} + \frac{a_8}{5} - \frac{a_9}{4} & 13 \\ \frac{a_7}{7} - \frac{a_8}{6} + \frac{a_9}{5} & 14 \\ \frac{a_8}{7} - \frac{a_9}{6} & 15 \\ \frac{a_9}{7} & 16 \end{pmatrix}$$

$$W_{4,0} \rightarrow 4\cdot a_3 - \frac{a_2}{2} + 30\cdot a_6 + \frac{1}{3}$$

Given

$W_{0,0} = 0 \qquad W_{1,0} = 1 \qquad W_{2,0} = 0$

$W_{3,0} = 0 \qquad W_{4,0} = 0 \qquad W_{5,0} = 0$

$$w := \mathrm{Find}(a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow \begin{pmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{6} \\ \frac{1}{40} \\ -\frac{1}{30} \\ \frac{1}{36} \end{pmatrix}$$

$$ys(x, k) := a_0 + a_1 \cdot x + \sum_{i=0}^{k-2} \left(w_i \cdot x^{i+2}\right)$$

$$ys(x, 7) \rightarrow \frac{x^7}{36} - \frac{x^6}{30} + \frac{x^5}{40} - \frac{x^4}{6} + \frac{x^3}{6} + x$$

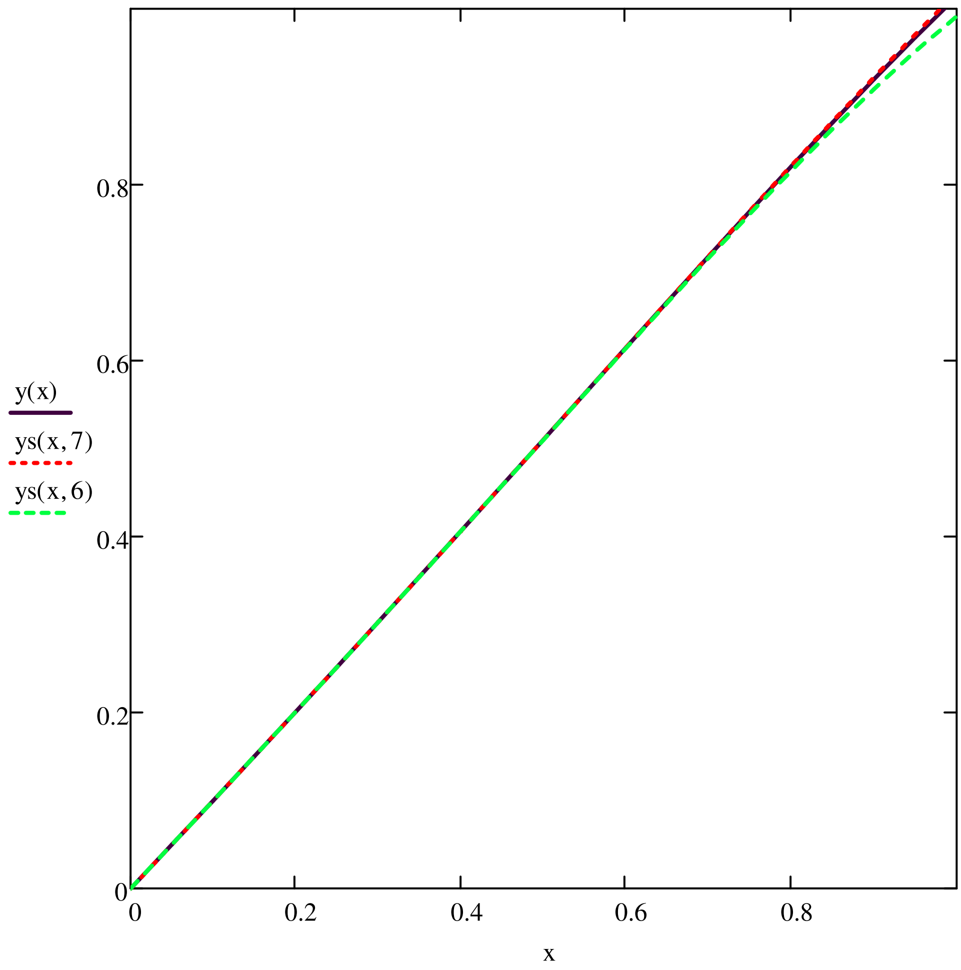
Given

$$y''(x) + x^2 \cdot y'(x) + \ln(1 + x)\, y(x) = x$$

$y(0) = 0 \qquad y'(0) = 1$

$y := \mathrm{Odesolve}(x, 1, 100)$

$x := 0, 0.05 .. 1$



$y(0.64) = 0.65718$

$ys(0.64, 7) = 0.65734$

$ys(0.64, 6) = 0.65612$

$ys(0.64, 5) = 0.65841$

Metoda szeregów potęgowych - przykład P2

Znaleźć rozwiązanie szczególne zagadnienia początkowego

$y'' + \cos(x)\cdot y' + e^{x}\cdot y = 0;\quad y(0) = 1;\quad y'(0) = 0$

w otoczeniu punktu $x = 0$

$$y(x) := a_0 + a_1\cdot x + a_2\cdot x^2 + a_3\cdot x^3 + a_4\cdot x^4 + a_5\cdot x^5 + a_6\cdot x^6 + a_7\cdot x^7 + a_8\cdot x^8 + a_9\cdot x^9$$

$$y'(x) := \frac{d}{dx}y(x) \rightarrow 9\cdot a_9\cdot x^8 + 8\cdot a_8\cdot x^7 + 7\cdot a_7\cdot x^6 + 6\cdot a_6\cdot x^5 + 5\cdot a_5\cdot x^4 + 4\cdot a_4\cdot x^3 + 3\cdot a_3\cdot x^2 + 2\cdot a_2\cdot x + a_1$$

$$y''(x) := \frac{d^2}{dx^2}y(x) \rightarrow 72\cdot a_9\cdot x^7 + 56\cdot a_8\cdot x^6 + 42\cdot a_7\cdot x^5 + 30\cdot a_6\cdot x^4 + 20\cdot a_5\cdot x^3 + 12\cdot a_4\cdot x^2 + 6\cdot a_3\cdot x + 2\cdot a_2$$

$$cs(x) := \cos(x)\ \text{series}, x, 7\ \rightarrow 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$$ex(x) := e^{x}\ \text{series}, x, 7\ \rightarrow 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$$

$$F(x) := y''(x) + cs(x)\cdot y'(x) + ex(x)\cdot y(x)$$

$$F(x)\ \mathrm{coeffs}, x, \mathrm{degree}\ \rightarrow \left(\begin{array}{cc}
a_0 + a_1 + 2\cdot a_2 & 0 \\
a_0 + a_1 + 2\cdot a_2 + 6\cdot a_3 & 1 \\
\frac{a_0}{2} + \frac{a_1}{2} + a_2 + 3\cdot a_3 + 12\cdot a_4 & 2 \\
\frac{a_0}{6} + \frac{a_1}{2} + a_3 + 4\cdot a_4 + 20\cdot a_5 & 3 \\
\frac{a_0}{24} + \frac{5\cdot a_1}{24} + \frac{a_2}{2} - \frac{a_3}{2} + a_4 + 5\cdot a_5 + 30\cdot a_6 & 4 \\
\frac{a_0}{120} + \frac{a_1}{24} + \frac{a_2}{4} + \frac{a_3}{2} - a_4 + a_5 + 6\cdot a_6 + 42\cdot a_7 & 5 \\
\frac{a_0}{720} + \frac{a_1}{144} + \frac{a_2}{24} + \frac{7\cdot a_3}{24} + \frac{a_4}{2} - \frac{3\cdot a_5}{2} + a_6 + 7\cdot a_7 + 56\cdot a_8 & 6 \\
\frac{a_1}{720} + \frac{a_2}{180} + \frac{a_3}{24} + \frac{a_4}{3} + \frac{a_5}{2} - 2\cdot a_6 + a_7 + 8\cdot a_8 + 72\cdot a_9 & 7 \\
\frac{a_2}{720} + \frac{a_3}{240} + \frac{a_4}{24} + \frac{3\cdot a_5}{8} + \frac{a_6}{2} - \frac{5\cdot a_7}{2} + a_8 + 9\cdot a_9 & 8 \\
\frac{a_3}{720} + \frac{a_4}{360} + \frac{a_5}{24} + \frac{5\cdot a_6}{12} + \frac{a_7}{2} - 3\cdot a_8 + a_9 & 9 \\
\frac{a_4}{720} + \frac{a_5}{720} + \frac{a_6}{24} + \frac{11\cdot a_7}{24} + \frac{a_8}{2} - \frac{7\cdot a_9}{2} & 10 \\
\frac{a_5}{} + \frac{a_7}{} + \frac{a_8}{} + \frac{a_9}{} & 11
\end{array}\right.$$

$$720 \quad 24 \quad 2 \quad 2$$

$$\begin{pmatrix} \frac{a_6}{720} - \frac{a_7}{720} + \frac{a_8}{24} + \frac{13 \cdot a_9}{24} & 12 \\ \frac{a_7}{720} - \frac{a_8}{360} + \frac{a_9}{24} & 13 \\ \frac{a_8}{720} - \frac{a_9}{240} & 14 \\ \frac{a_9}{720} & 15 \end{pmatrix}$$

Given

$$a_0 = 1$$

$$a_1 = 0$$

$$a_0 + a_1 + 2 \cdot a_2 = 0$$

$$a_1 + 2 \cdot a_2 + 6 \cdot a_3 + a_0 = 0$$

$$a_2 + 3 \cdot a_3 + 12 \cdot a_4 + \frac{1}{2} \cdot a_0 + \frac{1}{2} \cdot a_1 = 0$$

$$a_3 + 4 \cdot a_4 + 20 \cdot a_5 + \frac{1}{2} \cdot a_1 + \frac{1}{6} \cdot a_0 = 0$$

$$a_4 + 5\cdot a_5 + 30\cdot a_6 + \frac{5}{24}\cdot a_1 + \frac{1}{2}\cdot a_2 + \frac{1}{24}\cdot a_0 - \frac{1}{2}\cdot a_3 = 0$$

$$a_5 + 6\cdot a_6 + 42\cdot a_7 + \frac{1}{4}\cdot a_2 + \frac{1}{2}\cdot a_3 - a_4 + \frac{1}{120}\cdot a_0 + \frac{1}{24}\cdot a_1 = 0$$

$$\frac{a_0}{720} + \frac{a_1}{144} + \frac{a_2}{24} + \frac{7\cdot a_3}{24} + \frac{a_4}{2} - \frac{3\cdot a_5}{2} + a_6 + 7\cdot a_7 + 56\cdot a_8 = 0$$

$$a := \mathrm{Find}\left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\right) \rightarrow \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{120} \\ \frac{1}{120} \\ \frac{1}{560} \\ -\frac{1}{4032} \end{pmatrix}$$

$k := 8$

$$ys(x) := \sum_{i=0}^{k} \left(a_i \cdot x^i\right) \qquad ys(x) \rightarrow \frac{x^7}{560} - \frac{x^8}{4032} + \frac{x^6}{120} - \frac{x^5}{120} - \frac{x^2}{2} + 1$$

Given

$$y''(x) + \cos(x) \cdot y'(x) + e^x y(x) = 0$$

$$y(0) = 1$$

$$y'(0) = 0$$

$$y := \mathrm{Odesolve}(x, 2.6)$$

$$x := 0, 0.1 .. 2.6$$