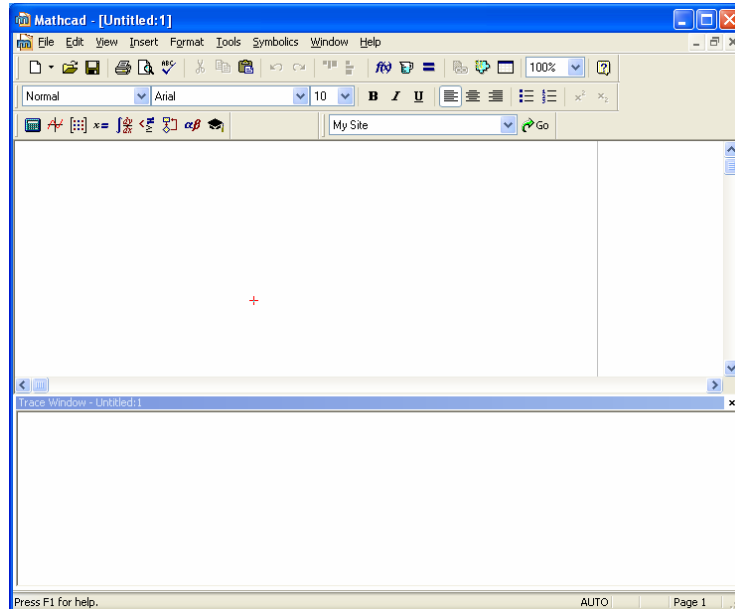


Introduction to *Mathcad*

Prepared by Gilberto E. Urroz, February 2006

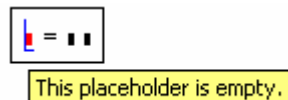
To get started, find the *Mathcad 13* icon in your *Start>Programs* button in Windows. The *Mathcad* interface is shown below.



The interface shows a main window and a *Trace window*. At this point we will not use the trace window, therefore, you could click it off leaving only the main window for operations.

Using Mathcad as a calculator

Mathcad can be used as a calculator. For example, click on the main window in an area near the top left corner, and type an equal sign (=). This produces the following entry in the worksheet:



The placeholder to the left of the equal sign (in red) is waiting for an entry. Start typing:


$$1+2/3+4/5$$

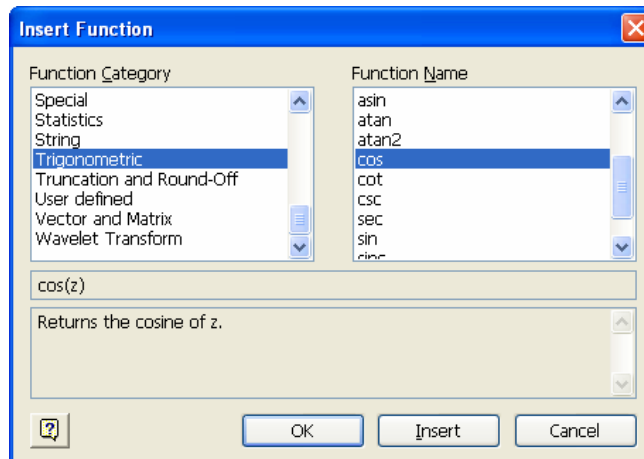
Then, click somewhere else in the main window. The result is the following:

$$1 + \frac{2}{3 + \frac{4}{5}} = 1.526$$

This exercise illustrates the use of Mathcad to perform simple calculations.

Functions in Mathcad

Calculations may involve mathematical functions such as *sin*, *cos*, *exp*, etc. You can insert any function by using the $f(x)$ button in the toolbar . Pressing this button provides a menu of functions. The figure below illustrates the use of the function *cosine*:



As an alternative, you can simply type the name of the function, i.e., *cos*.

As an exercise, click in another area of the main *Mathcad* window, type an equal sign, and in the red placeholder type the following expression (you can either use the $f(x)$ button or type the name of the functions *sin* and *cos*):

$$2.5*\sin(2.5)+1.2*\cos(2+3/(1+4*1.2^2))$$

Then, click somewhere else in the main window. The result of this operation is now shown as:

$$2.5 \cdot \sin(2.5) + 1.2 \cdot \cos\left[2 + \frac{3}{(1 + 4 \cdot 1.2^2)}\right] = 0.577$$

It is not always necessary to type the equal sign first to calculate a numerical expression. For example, click somewhere in the worksheet and type the following:

$$(2.5+3.2/(1.2+\sin(1.25)))=$$

The result is:

$$\left[2.5 + \frac{3.2}{(1.2 + \sin(1.25))}\right] = 3.989$$

Using Mathcad function root to find the root of an equation

Explore on your own the functions available under the $f(x)$ button. Some functions, such as \sin and \cos , are straightforward mathematical functions, while others are *Mathcad* functions used to perform specific operations in the worksheet. As an example, try the following exercise:

1. Click in an area of the main *Mathcad* window
2. Type an equal sign (=)
3. Using the $f(x)$ button, select function *root* under the *Solving* category. Press the [Ok] button. The result is the following: $\text{root}(\cdot, \cdot, \cdot, \cdot)$
4. Click on the first placeholder and type x^3 [\rightarrow] $-x-24$
5. Click on the second placeholder and type x
6. Click on the third placeholder and type 0
7. Click on the fourth placeholder and type 5
8. Click somewhere else in the worksheet to see the result:

$$\text{root}(x^3 - x - 24, x, 0, 5) = 3$$

Function *root* was used to find the solution of the equation $x^3 - x - 24 = 0$, specifying that the unknown of interest is x , and providing the interval $0 < x < 5$ as the preferred interval for the solution. The right-hand side of this result shows that the root sought is $x = 3$.

Defining variables in Mathcad

To define a variable in *Mathcad* use the *assignment operator*, namely, a *colon* followed by an equal sign $:=$, which you can get by typing a colon ($:$). Click anywhere in the worksheet and make the following variable assignment: $x := -2$. Then, click somewhere else in the worksheet. The value of -2 is now stored in the name x . The value $x = -2$ will be replaced into any expression containing the name x that is located below or to the right of the assignment statement. To check this fact, type an equal sign in a location above the assignment statement $x := -2$ and fill the placeholder to the left with the expression $x^2 - 3$. The result is inconclusive, i.e.,

$$x^2 - 3 = \blacksquare$$

Then, repeat this operation in a location below or to the right of the assignment statement $x := -2$. The result now is:

$$x^2 - 3 = 5$$

Once you have made variable assignments, you can use the assigned variables to calculate expressions. As an example, assign the values $x := -2$ and $y := 3$, then calculate the following expression: $x^2 + y^2$. The result may look somewhat like this:

$$x := -2 \quad y := 3$$

$$x^2 + y^2 = 13$$

Documenting the worksheet

In order to document your worksheet, you can add text to it by clicking on any place in the worksheet and typing " followed by the text. (Alternatively, you can use *Insert>Text Region* in the worksheet menu). For example, the last exercise above, can be documented as follows:

For the values $x := -2$ and $y := 3$
the sum of squares is calculated as: $x^2 + y^2 = 13$

Selecting fields in the worksheet for editing and repositioning

If you click on any of the text lines or the operations shown above, the corresponding field will be shown enclosed in a rectangle. This indicates that the field can be edited for text or math calculations. An example is shown below:

For the values $x := -2$ and $y := 3$
the sum of squares is calculated as: $x^2 + y^2 = 13$

A text or calculation field can also be selected by clicking in a region near the field and dragging the mouse towards the field. In this case, the field is enclosed by a frame with a broken line for boundary, e.g.,

For the values $x := -2$ and $y := 3$
the sum of squares is calculated as: $x^2 + y^2 = 13$

When selected in this way, a field can be dragged and positioned somewhere else in the worksheet. For example, the result shown above could be re-organized as follows:

For the values $x := -2$ and $y := 3$
the sum of squares is calculated as:

$$x^2 + y^2 = 13$$

A field selected by dragging can be erased from the worksheet by pressing the *[delete]* or the *[backspace]* key. The selected field can also be copied and pasted by using *Cntl-C* and *Cntl-V*, respectively. By dragging on top of two or more fields, multiple fields can be selected for deletion, copying, or relocation.

Equation solution using variables

In an earlier section we introduced function *root* for the solution of equations. The example presented earlier required four arguments, namely, the left-hand side of the equation, the unknown, and the limits of the interval where the solution was sought.

Using variables, we can provide an initial guess for the unknown, and then use function *root* with only the first two arguments to solve the equation. The following example repeats the solution shown earlier, providing an initial guess through a variable assignment:

$$x := -100 \quad \text{root}(x^3 - x - 24, x) = 3$$

Solution of a single equation using a *Given-Find* block

Suppose that you want to solve an equation of the form $f(\dots, x, \dots) = g(\dots, x, \dots)$ for x . The solution can be accomplished by using a *Given-Find* block according to the following structure:

$x := x0$ Given $f(\dots, x, \dots) = g(\dots, x, \dots)$ $x := \text{Find}(x) \quad x =$
--

where $x := x0$ is an initial guess for the solution, and the symbol $=$ in the equation is entered by pressing *Ctrl =*. Notice that the initial guess is shown before the *Given-Find* block, and that the solution requires the use of function *Find* after the equation is defined. To see the solution to the equation, we add the expression $x =$ to the right of the *Find* statement. The field $x =$ will show the actual value of the solution. Consider the following example:

$$x := -2$$

Given

$$x^2 - 5 \cdot x + 6 = 0$$

$$x := \text{Find}(x) \quad x = 2$$


To find a different real solution for the equation (if it exists) start with a different initial guess. For this case, for example, we change the initial guess from -2 to, say, 5. *Mathcad* then recalculates the solution automatically. The new result is shown below:

$$x := 5$$

Given

$$x^2 - 5 \cdot x + 6 = 0$$

$$x := \text{Find}(x) \quad x = 3$$


Note: If you're not sure that *Mathcad* has recalculated your solution, you can force a recalculation by pressing the *Calculate* button in the toolbar .

Solving multiple equations with a *Given-Find* block

The solution of multiple equations requires defining all the equations after the *Given* particle in the block, and using a vector to which function *Find* is assigned. Here is an example:

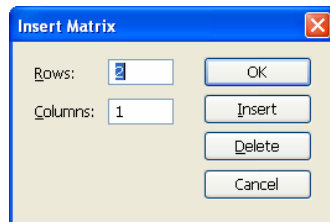
$$\begin{array}{l}
 x := 2 \quad y := 3 \\
 \text{Given} \\
 x^2 + y^2 = 25 \\
 x^3 + y^3 = 91 \\
 \begin{pmatrix} x \\ y \end{pmatrix} := \text{Find}(x, y) \\
 x = 3 \quad y = 4
 \end{array}$$

Notes:

1. Initial guesses for the unknowns ($x:=2, y:=3$) are required before the *Given* block.
2. Remember to use *Ctrl* = to generate the symbol = in the equations.
3. To enter the vector to which function *Find* is assigned, use the *Vector and Matrix toolbar* button in the toolbar . This generates the following palette:



Press the *Matrix and vector* button , to generate the following entry form:



To enter the vector required in the example above, select 2 rows and 1 column. Then type the variable names x and y in the corresponding placeholders. [As an alternative way to enter a vector or matrix, use *Ctrl*+*M* after selecting an entry location in the worksheet].

4. Type $x =$ and $y =$ after function *Find* to see the results for x and y .

The approach presented here for 2 equations can be generalized to any number of equations. In the following example, we use a *Given-Find* block to solve a problem in hydraulics.

W.R.E. 12-18. A pump is installed to deliver water from a reservoir of surface elevation zero to another of elevation 300 ft. The 12-in-diameter suction pipe ($f = 0.020$) is 100 ft long and the 10-in-diameter discharge pipe ($f = 0.026$) is 5000 ft long. The pump characteristic at 1200 rpm is defined by $H = 375 - 24 Q^2$ where H , the pump head is in feet and Q is in cubic feet per second. Compute the rate at which this pump will deliver water under these conditions assuming the setting is low enough to avoid cavitation.

$$\text{cfs} := \text{ft}^3 \cdot \text{sec}^{-1}$$

Energy losses are written as:
$$h_f = f \cdot \frac{L}{D} \cdot \frac{v^2}{2 \cdot g} \quad \text{with,} \quad v = \frac{4 \cdot Q}{\pi \cdot D^2}, \quad \frac{v^2}{2 \cdot g} = \frac{8 \cdot Q^2}{\pi^2 \cdot g \cdot D^4}$$

and,

$$h_f = \frac{8 \cdot f \cdot L}{\pi^2 \cdot g \cdot D^5} \cdot Q^2$$

The system equation will be:

$$H = \Delta H + \frac{8 \cdot f_1 \cdot L_1}{\pi^2 \cdot g \cdot D_1^5} \cdot Q^2 + \frac{8 \cdot f_2 \cdot L_2}{\pi^2 \cdot g \cdot D_2^5} \cdot Q^2$$

with, $f_1 := 0.020$ $L_1 := 100 \cdot \text{ft}$ $D_1 := 12 \cdot \text{in}$ $f_2 := 0.026$ $L_2 := 5000 \cdot \text{ft}$ $D_2 := 10 \cdot \text{in}$

$$\Delta H := 300 \cdot \text{ft}$$

We'll use a Given-Find block to solve for (Q, H) from the system and pump equations:

First guesses: $Q := 1 \cdot \text{cfs}$ $H := 100 \cdot \text{ft}$

Given

$$H = 375.0 \cdot \text{ft} - \left(24 \cdot \frac{\text{ft}}{\text{cfs}^2} \right) \cdot Q^2$$

$$H = \Delta H + \frac{8 \cdot f_1 \cdot L_1}{\pi^2 \cdot g \cdot D_1^5} \cdot Q^2 + \frac{8 \cdot f_2 \cdot L_2}{\pi^2 \cdot g \cdot D_2^5} \cdot Q^2$$

$$\begin{pmatrix} Q \\ H \end{pmatrix} := \text{Find}(Q, H)$$

$$H = 319.099 \text{ ft}$$


$$Q = 1.526 \text{ s}^{-1} \text{ ft}^3$$

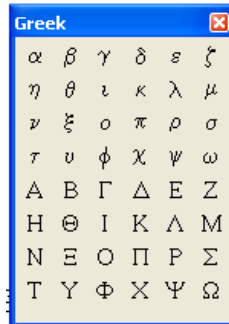
Notes:

[1] - Use *Format > Properties* and select *Highlight Region* in the *Display* tab to highlight a region in the worksheet. Press the [OK] button when done. Highlighting could be used to emphasize solutions in a worksheet as shown above.

[2] - This example illustrates the use of units in the calculations. Units usage is discussed in a subsequent section.

Using Greek letters in expressions

A palette with Greek letters is available by pressing the $\alpha\beta$ button in the toolbar . The resulting floating palette is shown below:



Clicking on any of the letters will copy that letter to any entry point or text in the worksheet. To illustrate this situation we perform a calculation with trigonometric functions that features the value π .

$$\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) = 1.366$$

Notice that *Mathcad* assigns the proper value to the symbol π in the calculation. [Note: The arguments of trigonometric functions in *Mathcad* need to be in radians. To convert an angle in degrees (θ°) to its equivalent in radians (θ^r), use: $\theta^r = \frac{\pi}{180} \cdot \theta^\circ$].

Besides π , there are other Greek letters that have special meanings:


- δ : calculates *Kronecker's delta function*, $\delta(i,j) = 1$ if $i=j$, or $\delta(i,j) = 0$ if $i \neq j$. δ can also be used to represent the impulse function $\delta(x)$.
- ϵ : calculates the *third-order epsilon tensor*, e.g., $\epsilon(1,0,2) = -1$. This tensor is used in the definition of determinants for 3×3 matrices.
- Γ : calculates the *Gamma function*, e.g., $\Gamma(2.3) = 1.167$. This function is related to factorials of integer numbers as follows: $\Gamma(n+1) = n!$. The *Gamma* function is also used to define probability distributions of continuous random variables.
- Φ : calculates *Heaviside step function*, $\Phi(x) = 0$ if $x < 0$, $\Phi(x) = 1$ if $x > 0$.

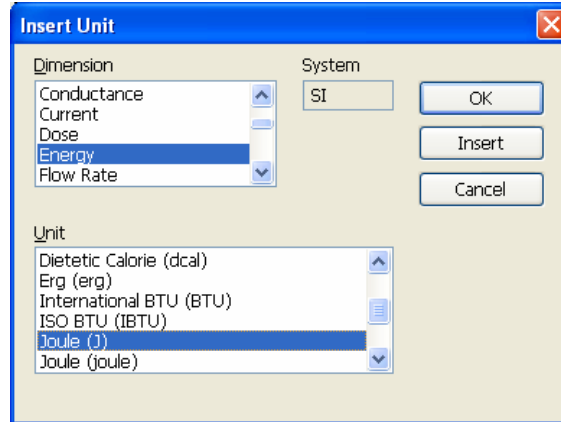
An alternative way to enter Greek letters is to type the equivalent English letter and then type *Cntl G*. Try this exercise, after clicking on an entry location in the worksheet:

$$a \text{ Cntl } G := \sin(60 * p \text{ Cntl } G / 180) + \cos(75 * p \text{ Cntl } G / 180)$$

Verify that the value of α is 1.125. (Click in a location below the equation above, and type *a Cntl G =*).

Calculations using units

Units can be attached to numerical values by using the units button in the toolbar . Pressing the button provides a menu of units as shown below (here, the *SI* unit of energy, the *joule*, is selected).



Pressing the [OK] button at this point would attach the *joule* units to a numerical value. As an exercise, define, in your worksheet, the following variables with units, and calculate the expression shown for v :

$$v_0 := 2.5 \cdot \frac{\text{m}}{\text{s}} \quad a := -1.2 \cdot \frac{\text{m}}{\text{s}^2} \quad t := 0.75 \cdot \text{s}$$

$$v := v_0 + a \cdot t \quad v = 1.6 \frac{\text{m}}{\text{s}}$$

Notes:

1. To attach units use the symbol * between the numerical value and the unit. For example, type $v_0 := 2.5 * \text{m/s}$, then click somewhere else in the worksheet.
2. Although there is category for units of *Velocity*, the unit m/s is not included because it can be put together from m (in the *Length* category) and s (in the *Time* category). As an alternative, you can simply type the unit combination: m/s .
3. The calculation shown above uses five fields. The three fields in the first row were used to define variables with units. The first field in the second row represents the calculation of the variable $v := v_0 + at$. However, this operation alone would not show the value of v . For that we need to add the last field in the second row, i.e., type $v =$

By default, a calculation involving units, *Mathcad* will provide the final result in units of the *SI* (*Syteme Internationale*), e.g.,

$$v_0 := 2.5 \cdot \frac{\text{ft}}{\text{s}} \quad a := -1.2 \cdot \frac{\text{m}}{\text{s}^2} \quad t := 0.75 \cdot \text{s}$$

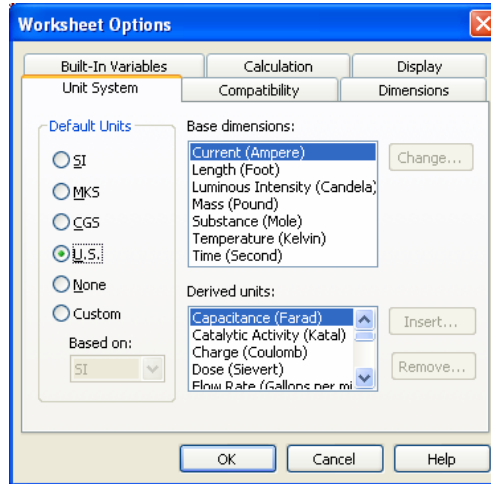
$$v := v_0 + a \cdot t \quad v = -0.138 \frac{\text{m}}{\text{s}}$$

Another example:

$$v_0 := 2.5 \cdot \frac{\text{ft}}{\text{s}} \quad a := -1.2 \cdot \frac{\text{in}}{\text{sec}^2} \quad t := 0.75 \cdot \text{sec}$$

$$v := v_0 + a \cdot t \quad v = 0.739 \frac{\text{m}}{\text{s}}$$

To change the default unit system, select *Tools>Worksheet Options*, and click on the *Unit system* tab (see below).



The units systems available include *MKS* (*meter-kilogram-second*), *CGS* (*centimeter-gram-second*), and *US* (i.e., U.S. customary, or English, system). For example, selecting *US* as the default *Unit system* produces the following result for the last example above:

$$v_0 := 2.5 \cdot \frac{\text{ft}}{\text{s}} \quad a := -1.2 \cdot \frac{\text{in}}{\text{sec}^2} \quad t := 0.75 \cdot \text{sec}$$

$$v := v_0 + a \cdot t \quad v = 2.425 \frac{\text{ft}}{\text{s}}$$

After a result has been calculated, you can do a unit conversion by replacing the units in the result with an equivalent unit. For example, the following calculations (with *US* as the default *Unit system*), provide a result in *psi* (pounds per square inch):

$$F := 230 \cdot \text{lbf} \quad A := 23 \cdot \text{in}^2$$

$$p := \frac{F}{A} \quad p = 10 \text{ psi}$$

If you click on the *psi* units, a placeholder appears to the right of *psi*. Click on the placeholder and type *psf* (pounds per square foot) to convert the result to those units, i.e.,

$$F := 230 \cdot \text{lbf} \quad A := 23 \cdot \text{in}^2$$

$$p := \frac{F}{A} \quad p = 1.44 \times 10^3 \text{ psf}$$

It is possible to define your own units as in the following example, in which the unit *cfs* (cubic feet per second) is defined by the user. The result, provided by *Mathcad*, is given in *gpm* (gallons per minute):

$$\text{cfs} := \frac{\text{ft}^3}{\text{sec}}$$

$$\underline{V} := 2.45 \cdot \frac{\text{ft}}{\text{sec}} \quad \underline{A} := 1.2 \cdot \text{ft}^2 \quad Q := V \cdot A \quad Q = 1.32 \times 10^3 \text{ gpm}$$

By clicking on the *gpm* units and replacing the placeholder with *cfs* we get:

$$\underline{V} := 2.45 \cdot \frac{\text{ft}}{\text{sec}} \quad \underline{A} := 1.2 \cdot \text{ft}^2 \quad Q := V \cdot A \quad Q = 2.94 \text{ cfs}$$

Saving the worksheet

To save the worksheet, use *File>Save As ...* and give it a name (e.g., *FirstWorksheet*). The file will be given the sub-index *xmcd*. Older versions of *Mathcad* used *mcd*.

To create a new worksheet, use *Cntl-n*. Here is an example of a well-documented *Mathcad* worksheet:

CEE 3500 - Spring 2006 - Problem using continuity and energy equation in a pipeline.

A pipeline with a 3-in diameter is connected to a 6-in pipeline through a smooth transition. A gage located in the 3-in pipeline, at an elevation of 2 ft (point A), reads a pressure of 5 psi, while a gage in the 6-in pipeline, at an elevation of 3.5 ft (point B), reads 2 psi. Determine the energy losses between points A and B if the discharge through the pipeline is 10 gpm.

Solution The data given can be interpreted as follows:

DA := 3-in DB := 6-in zA := 2-ft pA := 5-psi zB := 3.5-ft pB := 2-psi

Q := 10-gpm $\gamma := 62.4 \frac{\text{lb}_f}{\text{ft}^3}$

The flow velocities are calculated as follows:

$V_A := \frac{4Q}{\pi \cdot D_A^2}$ $V_A = 0.454 \frac{\text{ft}}{\text{s}}$ $V_B := \frac{4Q}{\pi \cdot D_B^2}$ $V_B = 0.113 \frac{\text{ft}}{\text{s}}$

The solution of the energy equation can be accomplished using a *Given-Find* block as illustrated next.

Initial guess for head losses *hL*: hL := 0.5-ft

Given

$$\left(\frac{p_A}{\gamma} \right) + z_A + \left(\frac{V_A^2}{2 \cdot g} \right) - hL = \left(\frac{p_B}{\gamma} \right) + z_B + \left(\frac{V_B^2}{2 \cdot g} \right)$$

$\underline{hL} := \text{Find}(hL)$ hL = 5.426 ft

Here is another example of a *Mathcad* worksheet showing the solution to a compressible flow problem:

21.15. Air flow at a velocity of 121.9 m/s occurs in a duct. If the temperature and pressure of this air flow are 20°C and 96.5 kPa abs, respectively, determine the following:

(a) the local sonic velocity (use equation 4-11, p 23)

$$\text{kPa} := 1000 \cdot \text{Pa}$$

$$k := 1.4 \quad T_1 := (273.15 + 20) \cdot \text{K} \quad R := 287 \cdot \left(\frac{\text{newton} \cdot \text{m}}{\text{kg} \cdot \text{K}} \right) \quad V_1 := 121.9 \cdot \frac{\text{m}}{\text{sec}}$$

$$c := \sqrt{k \cdot R \cdot T} \quad c = 343.202 \text{ m} \cdot \text{s}^{-1} \quad M := \frac{V}{c} \quad M = 0.355 \quad p := 96.5 \cdot \text{kPa}$$

$$\rho := \frac{p}{R \cdot T} \quad \rho = 1.147 \text{ kg} \cdot \text{m}^{-3} \quad c_p := 1005 \cdot \frac{\text{newton} \cdot \text{m}}{\text{kg} \cdot \text{K}}$$

(b) the sonic velocity c at a point where this flow is changed isentropically to a Mach number of 1 (this sonic velocity is commonly used as a reference in tables and equations in gas dynamics)

$$p_1 := 96.5 \cdot \text{kPa} \quad p_2 := p_1 \left(\frac{2}{k+1} \right)^{\frac{k}{k-1}} \quad p_2 = 50.979 \text{ kPa} \quad \rho_1 := \rho$$

$$V_2 := \left[2 \cdot \frac{p_1}{\rho} \cdot \frac{k}{k-1} \cdot \left[1 - \left(\frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right] \right]^{\frac{1}{2}} \quad V_2 = 313.299 \text{ m} \cdot \text{s}^{-1}$$

(c) the sonic velocity c_0 at a stagnation point in this flow. (This sonic velocity is also a reference).

$$T_1 := T \quad T_1 = 293.15 \text{ K} \quad p_1 = 96.5 \text{ kPa} \quad V_1 := V \quad V_1 = 121.9 \text{ m} \cdot \text{s}^{-1}$$

We need to solve the following equations simultaneously; Assume: $p_0 := 50 \cdot \text{kPa}$ $T_0 := 200 \cdot \text{K}$

Given

$$\frac{p_0}{p_1} = \left(\frac{T_0}{T_1} \right)^{\frac{k}{k-1}} \quad \frac{V_1^2}{2} = c_p \cdot T_0 \cdot \left[1 - \left(\frac{p_1}{p_0} \right)^{\frac{k-1}{k}} \right] \quad \left(\frac{p_0}{T_0} \right) := \text{Find}(p_0, T_0)$$

$$p_0 = 105.289 \text{ kPa} \quad T_0 = 300.543 \text{ K}$$

$$c_0 := \sqrt{k \cdot R \cdot T_0} \quad c_0 = 347.503 \text{ m} \cdot \text{s}^{-1}$$

Defining, tabulating, and plotting a function

A function of a single variable can be defined by using, for example, $f(x) = \text{expression}$. An example is shown below. After defining the function we evaluate it at $x = 3.5$, and use it to find its roots with *Mathcad* function *root* as shown. After finding the roots, the function is evaluated at the roots to verify that $f(x) = 0$ at those points:

$$f(x) := x^2 - 5x + 6$$

$$f(3.5) = 0.75$$

$$\text{root}(f(x), x, 2.5, 4) = 3 \quad f(3) = 0$$

$$\text{root}(f(x), x, 0, 2.5) = 2 \quad f(2) = 0$$

To produce a table of the function we need to define a vector of values of the independent variable x in a certain range $a < x < b$. Suppose that we want to divide the interval $[a, b]$ into n sub-intervals of the same width $\Delta x = (b-a)/n$, then we can define the vector of values of x using the rule $x_j = a + j \Delta x$, for $j = 0, 2, \dots, n$. [Check that $a = x_0$ and $b = x_n$]. Subsequently we can produce a vector of values of $y_j = f(x_j)$. To show the table of y -vs- x values, create a matrix (e.g. Z) and define x as its first column, and y as its second column. The type $Z = .$ A table of values of the function defined above is shown next:


$$a := -2 \quad b := 5 \quad n := 20 \quad \Delta x := \frac{(b - a)}{n} \quad \Delta x = 0.35$$



$$j := 0..n \quad x_j := a + j \cdot \Delta x \quad y_j := f(x_j)$$

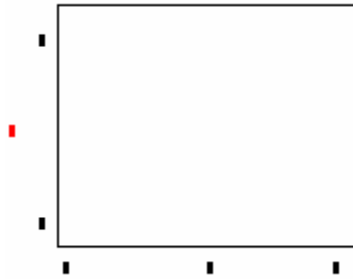
$$Z_{0,j} := x_j \quad Z_{1,j} := y_j$$

Z =		0	1	2	3	4	5	6	7	8
	0	-2.35	-2	-1.65	-1.3	-0.95	-0.6	-0.25	0.1	0.45
	1	23.273	20	16.973	14.19	11.653	9.36	7.313	5.51	3.953

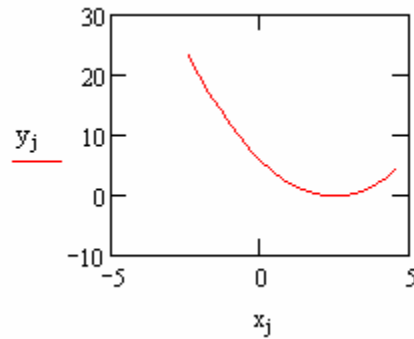
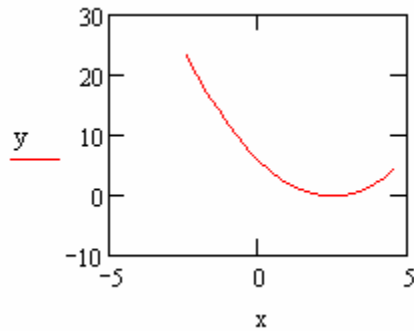
Notes:

1. To enter a range (e.g., $j := 0..n$), use a *semi-colon* between the range limits. For the example above, use: $j := 0;n$
2. To enter a sub-index, type [or click on the sub-index button in the toolbar .
3. If you prefer to see the table in the form of columns, change the assignment to matrix Z to the following: $Z_{j,0} := x_j$ and $Z_{j,1} := y_j$.

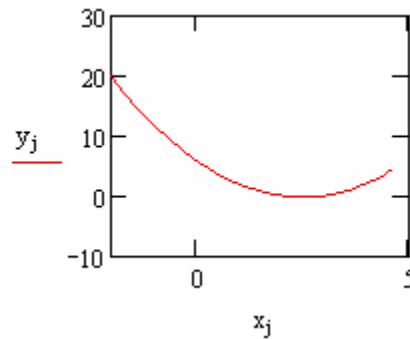
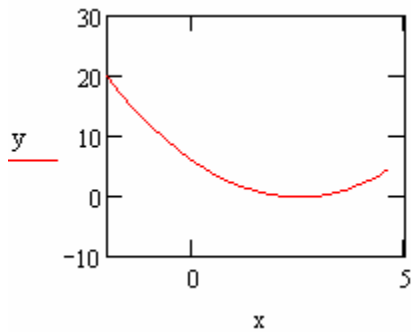
To produce a graph of the function, first, we need to generate the vectors of values of x and y , as we did above. Then, click on the region of the worksheet where you want the graph, and press the *Graph toolbar* button in the toolbar . From the resulting menu, select the *X-Y Plot* button . You will now have a graph frame with six placeholders, as shown in the following figure.



Replace the middle placeholders with x and y (or x_j and y_j) to produce one of the following graphs:



The graphs above use a default range for x , namely, $-5 < x < 5$. To change the range in a graph, click on the graph and replace the values shown at the ends of each axis. For example, in the figure below we change the range to $-2 < x < 5$:





The following example shows the development of a plot using polar coordinates. The function to plot is $r = g(\theta) = 5(1 - \cos(\theta))$. The function is defined, and values of θ and r are generated as follows:

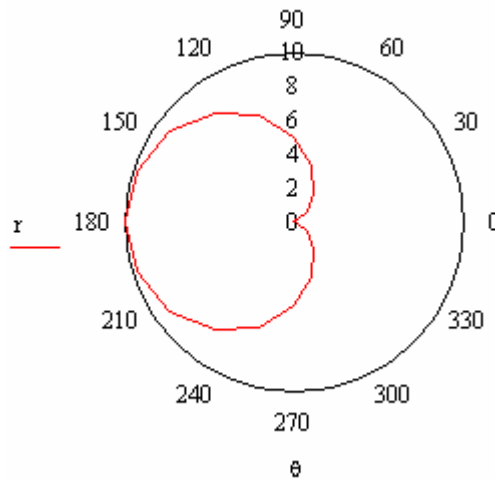
$$\begin{aligned}
 g(\theta) &:= 5 \cdot (1 - \cos(\theta)) & n &:= 20 \\
 \theta_0 &:= 0 & \theta_n &:= 2 \cdot \pi & \Delta\theta &:= \frac{(\theta_n - \theta_0)}{n} & \Delta\theta &= 0.314 \\
 j &:= 0..n+1 & \theta_j &:= \theta_0 + j \cdot \Delta\theta & r_j &:= g(\theta_j)
 \end{aligned}$$

To produce a table of the r - θ data, use a matrix T for this case:

$$T_{0,j} := \theta_j \quad T_{1,j} := r_j$$

	0	1	2	3	4	5	6	7	8	9	
T =	0	-0.314	0	0.314	0.628	0.942	1.257	1.571	1.885	2.199	2.513
	1	0.245	0	0.245	0.955	2.061	3.455	5	6.545	7.939	9.045

To select a polar plot, use the polar plot button  after clicking on the *Graph toolbar* button . The resulting plot is shown below, after replacing the names θ and r in the appropriate placeholders:



The resulting curve is known as a *cardioid* (i.e., heart-shaped).

Tables and plots of multiple functions

The approach shown above for tabulating and plotting a single function can be generalized to the tabulation and plotting of two or more functions. Try the following exercise:

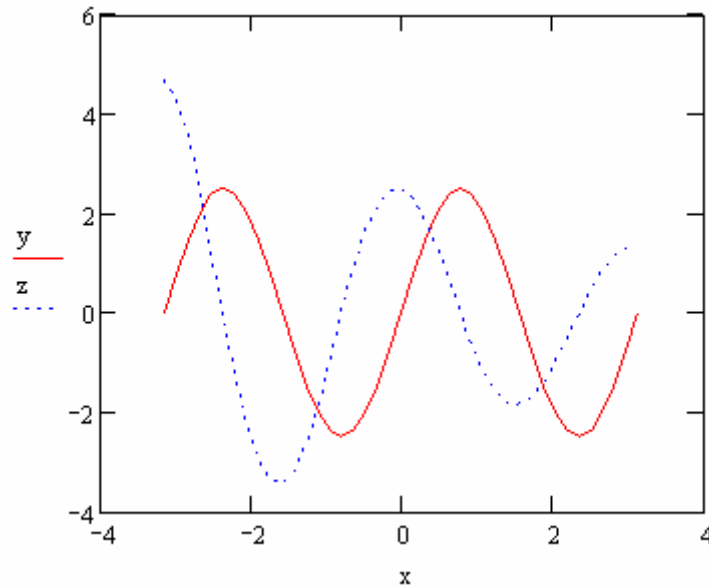
$$f(x) := 2.5 \cdot \sin(2 \cdot x) \quad g(x) := 2.5 \cdot \exp\left[-\left(\frac{x}{5}\right)\right] \cdot \cos(2x) \quad a := -\pi \quad b := \pi \quad n := 40$$

$$\Delta x := \frac{(b - a)}{n} \quad \Delta x = 0.157 \quad j := 0..n \quad x_j := a + j \cdot \Delta x \quad y_j := f(x_j) \quad z_j := g(x_j)$$

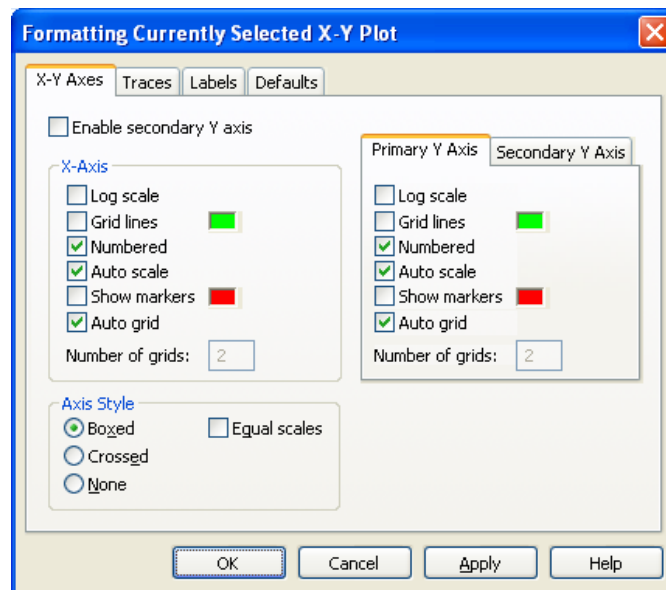
$$M_{0,j} := x_j \quad M_{1,j} := y_j \quad M_{2,j} := z_j$$

	0	1	2	3	4	5	6	7	8	9	
M =	0	-3.142	-2.985	-2.827	-2.67	-2.513	-2.356	-2.199	-2.042	-1.885	-1.728
	1	0	0.773	1.469	2.023	2.378	2.5	2.378	2.023	1.469	0.773
	2	4.686	4.319	3.56	2.507	1.277	0	-1.199	-2.211	-2.949	-3.359

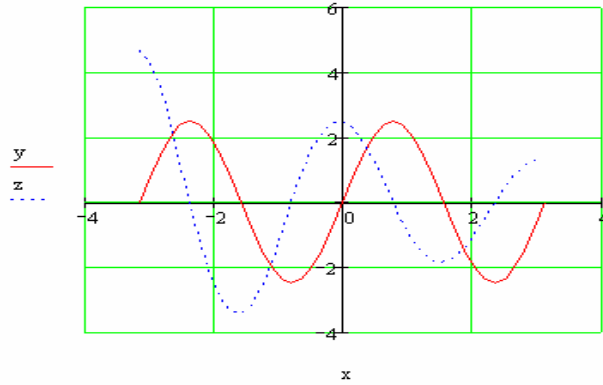
The plot of y -vs- x combined with that of z -vs- x is shown below. To produce this plot enter x in the horizontal axis (the abscissas) placeholder, and type y,z in the vertical axis (the ordinates) placeholder. The graph shown was enlarged by clicking on the graph and then dragging the lower right corner away from the center of the graph.



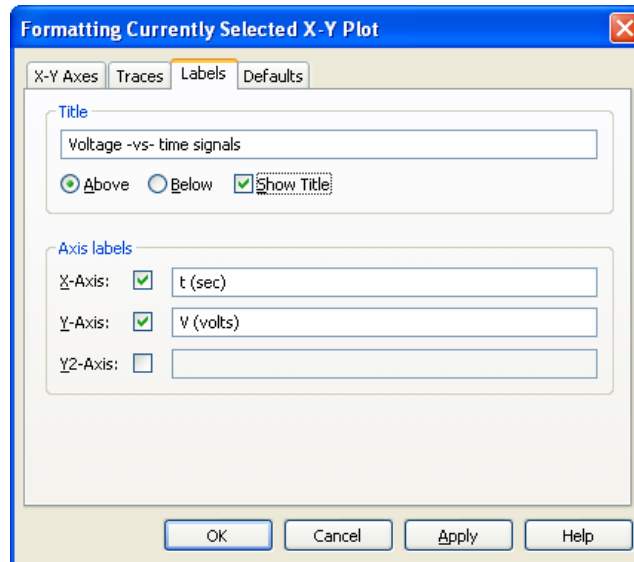
To change the properties of the graph, click on the graph and then do a right-click with the mouse. Select the entry *Format* from the resulting menu. The current (default) settings for axes properties are shown below:



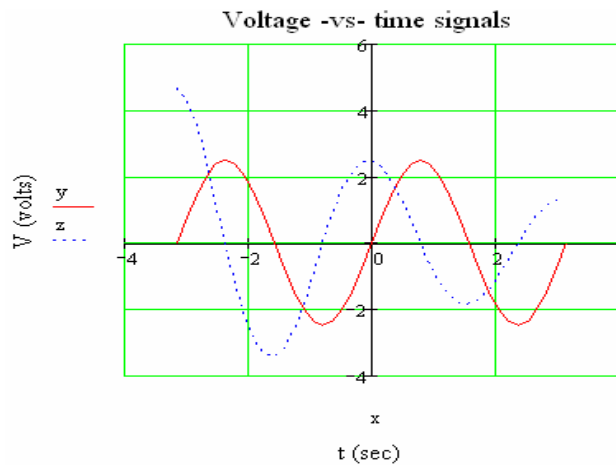
Select the [] *Grid lines* in both axes (*X-Axis* and *Primary Y Axis*), and select *Crossed* for the *Axis Style*. Press the [*Apply*] button to effect these changes in the graph. Then press the [*OK*] button. The modified graph is shown below:



Click on the graph once again, and do a right-click to select the *Format* window. This time, however, select the *Label* tab, and modified it to read as follows:



The fully-documented graph is shown below:



The following example illustrates the tabulations and plotting of two curves using a polar plot:

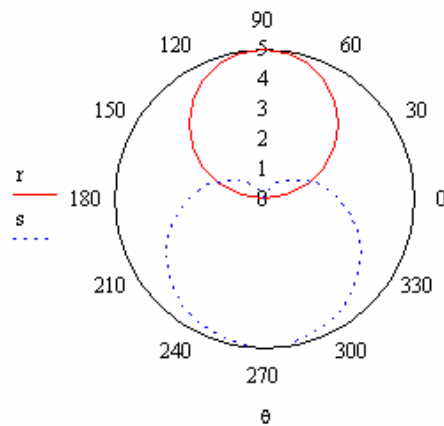
$$f(\theta) := 5 \cdot \sin(\theta) \quad g(\theta) := 2.5 \cdot (1 - \sin(\theta)) \quad a := -\pi \quad b := \pi \quad n := 40$$

$$\Delta x := \frac{(b - a)}{n} \quad \Delta x = 0.157 \quad j := 0..n \quad \theta_j := a + j \cdot \Delta x \quad r_j := f(\theta_j) \quad s_j := g(\theta_j)$$

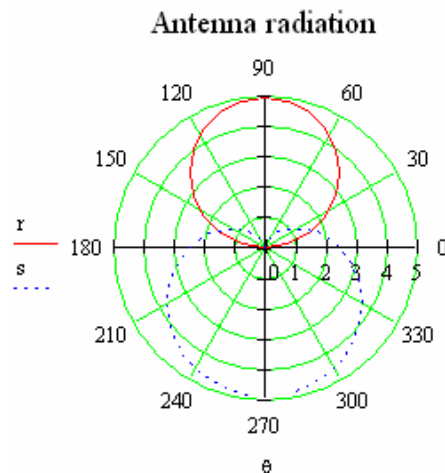
$$M_{0,j} := \theta_j \quad M_{1,j} := r_j \quad M_{2,j} := s_j$$

	0	1	2	3	4	5	6	7	8	9	
M =	0	-3.142	-2.985	-2.827	-2.67	-2.513	-2.356	-2.199	-2.042	-1.885	-1.728
	1	0	-0.782	-1.545	-2.27	-2.939	-3.536	-4.045	-4.455	-4.755	-4.938
	2	2.5	2.891	3.273	3.635	3.969	4.268	4.523	4.728	4.878	4.969

The graph is shown below:



Using the *Format* option from a right-click on the graph, select *Gridlines* for both the *Radial* and *Angular* coordinates, and *Crossed* for the *Axis style* in the *Polar Axes* tab. Also, in the *Labels* tab, type the following title: *Antenna radiation*. To modify the plot press the [Apply] and [OK] buttons. The resulting graph is shown below:



Multiply-defined functions

Suppose that we want to define the function

$$f(x) = \begin{cases} |x-1|, & \text{for } x < 1 \\ 1-x, & \text{for } x \geq 1 \end{cases}$$

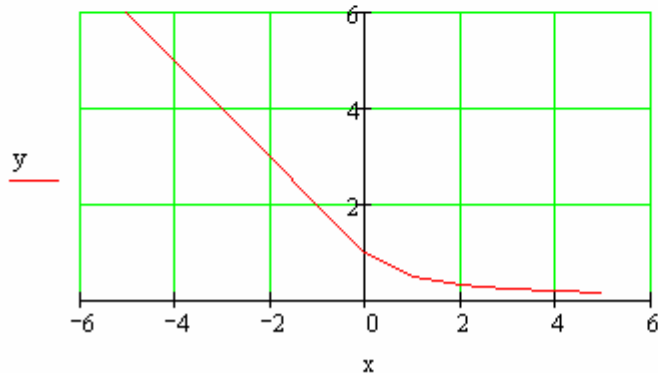
Mathcad provides function *if* for this purpose. The following exercise shows the definition of this function as well as a table and a plot of the same in the range $-5 < x < 5$:

$$f(x) := \text{if}\left(x < 1, |x-1|, \frac{1}{x+1}\right)$$

$$a := -5 \quad b := 5 \quad n := 10 \quad \Delta x := \frac{(b-a)}{n} \quad \Delta x = 1$$

$$j := 0..n \quad x_j := a + j \cdot \Delta x \quad y_j := f(x_j) \quad Z_{0,j} := x_j \quad Z_{1,j} := y_j$$

	0	1	2	3	4	5	6	7	8	9	
Z =	0	-5	-4	-3	-2	-1	0	1	2	3	4
	1	6	5	4	3	2	1	0.5	0.333	0.25	0.2



+

The *if* function has the general form *if(condition, option true, option false)*. If *condition* is true, then *option true* is selected. If *condition* is false, then *option false* is selected.

Consider now the following multiply-defined function that includes more than two possibilities:

$$f(x) = \begin{cases} x-1, & \text{for } x < 0 \\ x^2-1, & \text{for } 0 \leq x < 5 \\ 1-x^2, & \text{for } x \geq 5 \end{cases}$$

In this case it is possible to use a nested *if* function by re-defining the function statement as follows:

$$f(x) = \begin{cases} x-1, & \text{for } x < 0, \text{ exclusively} \\ x^2-1, & \text{for } 0 \leq x < 5, \text{ exclusively} \\ 1-x^2, & \text{otherwise} \end{cases}$$

Thus, using *Mathcad's* if function you can write:

$$\text{if}(x < 0, x-1, \text{if}(x < 5, x^2-1, 1-x^2))$$

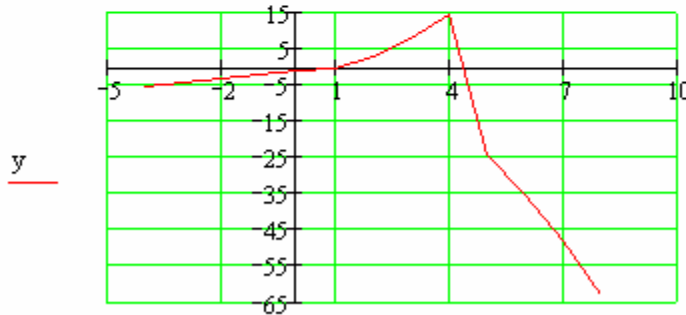
The following figure shows the implementation of this function in *Mathcad*, including a table and plot of the function:

$$f(x) := \text{if}(x < 0, x-1, \text{if}(x < 5, x^2-1, 1-x^2))$$

$$a := -4 \quad b := 8 \quad n := 12 \quad \Delta x := \frac{(b-a)}{n} \quad \Delta x = 1$$


$$j := 0..n \quad x_j := a + j \cdot \Delta x \quad y_j := f(x_j) \quad Z_{0,j} := x_j \quad Z_{1,j} := y_j$$

Z =	0	1	2	3	4	5	6	7	8	9	10	11	12
0	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
1	-5	-4	-3	-2	-1	0	3	8	15	-24	-35	-48	-63



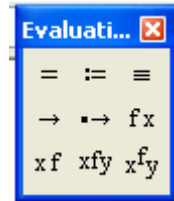
x

Symbolic evaluation of calculus operations

Use the *Calculus Toolbar* button in the toolbar  to produce the calculus toolbar:



The *Calculus Toolbar* includes buttons for limits, first- and higher-order derivatives, definite and indefinite integrals, summations, and products, as well as the infinite symbol. To calculate a symbolic (as opposite to a numerical) result, instead of using the equal sign we use the *symbolic evaluation* symbol \rightarrow , generated by using *Cntl .* (*control period*) in an expression. [Alternatively, the *symbolic evaluation* symbol (\rightarrow) can be selected from the *Evaluation Toolbar* (click the $x=$ button in the toolbar)]:



The *symbolic evaluation* symbol is the arrow located in the second row, first column of the *Evaluation Toolbar* shown above (\rightarrow).

The following example illustrates the symbolic evaluation of the derivative $\frac{d}{dx}(x^3 - x)$.

First, select the derivative button ($\frac{d}{dx}$) from the *Calculus Toolbar*. Then, type x in the lower placeholder and $(x^3 + x)$ in the higher placeholder. Finally, enter the *symbolic evaluation* symbol from the *Evaluation Toolbar*, or by typing *Cntl .* The result is shown below:

$$\frac{d}{dx}(x^3 - x) \rightarrow 3 \cdot x^2 - 1$$

Using the proper symbol from the *Calculus Toolbar* and the *symbolic evaluation* symbol (\rightarrow) evaluate the following calculus operations:

- A limit that defines a derivative

$$\lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin(x)}{h} \rightarrow \cos(x)$$

- A third-order derivative

$$\frac{d^3}{dx^3}(\exp(-x) \cdot \cos(x)) \rightarrow (-2) \cdot e^{-x} \cdot \sin(x) + 2 \cdot e^{-x} \cdot \cos(x)$$

- An infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \rightarrow \frac{1}{6} \cdot \pi^2$$

- A symbolic product

$$\prod_{r=1}^n x^r \rightarrow x^{\frac{1}{2} \cdot (n+1) \cdot n}$$

- An indefinite integral


$$\int \frac{1}{1+s^2} ds \rightarrow \text{atan}(s)$$

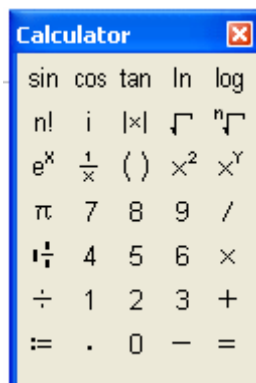
- A definite integral with an infinite limit

$$\int_0^{\infty} \exp\left[-\left(\frac{x^2}{2}\right)\right] dx \rightarrow \frac{1}{2} \cdot 2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}}$$

- A definite integral with symbolic limits

$$\int_{\alpha}^{\beta} \frac{1}{\sqrt{\xi}} d\xi \rightarrow 2 \cdot \beta^{\frac{1}{2}} - 2 \cdot \alpha^{\frac{1}{2}}$$

In the last expression we borrowed the *square-root symbol* ($\sqrt{\quad}$) from the *Calculator Toolbar*, which is activated by pressing the *Calculator Toolbar* button in the toolbar . The *Calculator Toolbar* is shown below. It includes symbols for trigonometric, logarithmic, and exponential functions, square root, any root, inverse, square, power, π , factorial, i (the unit imaginary number), improper fraction ($\frac{\square}{\square}$), the 10 digits, parentheses, and the four fundamental arithmetic operations.



Numerical evaluation of summations, products, and integrals

Summations, products, and integrals with numerical limits can be evaluated by *Mathcad* if we use the equal sign after the corresponding operation. Some examples are shown next. In each example we show both the numerical and the symbolic evaluation:

$$\sum_{k=1}^{10} \frac{1}{k^2} = 1.55$$

$$\sum_{k=1}^{10} \frac{1}{k^2} \rightarrow \frac{1968329}{1270080}$$

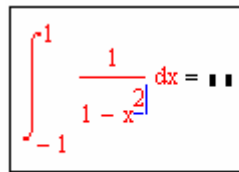
$$\prod_{s=2}^5 [s \cdot (s - 1)] = 2.88 \times 10^3$$

$$\prod_{s=2}^5 [s \cdot (s - 1)] \rightarrow 2880$$

$$\int_0^2 \frac{1}{1+x^2} dx = 1.107$$

$$\int_0^2 \frac{1}{1+x^2} dx \rightarrow \text{atan}(2)$$

In all the examples above the value of the operation converges to a finite result. If the operation diverges, a numerical evaluation would be inconclusive and *Mathcad* will provide a *no-convergence* message. The symbolic evaluation shows the result being infinite:



$$\int_{-1}^1 \frac{1}{1-x^2} dx \rightarrow \infty$$

This calculation does not converge to a solution.

Calculus operations on user-defined functions

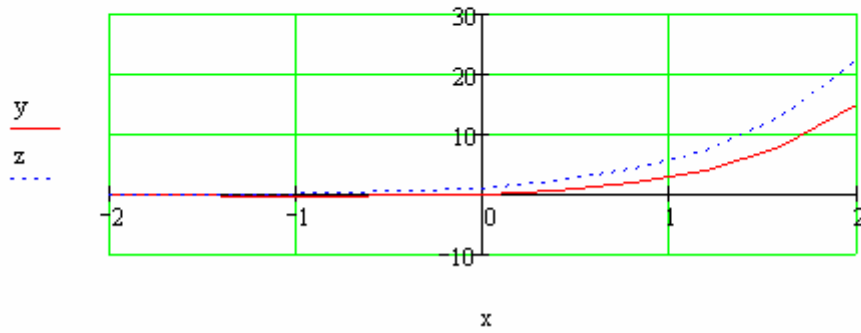
Users can define functions in *Mathcad* and subsequently use the operations in the *Calculus Toolbar* to calculate limits, derivatives, summations, products, and integrals. Other functions can also be defined based on those calculus operations of user-defined functions. In the following example a user-defined function $f(x)$ is used to define a second function $fp(x) = df/dx$, and tables and plots of the two functions are produced:

$$f(x) := x \cdot \exp(x) \quad fp(x) := \frac{d}{dx} f(x) \quad a := -2 \quad b := 2 \quad n := 10 \quad \Delta x := \frac{b-a}{n} \quad \Delta x = 0.4$$

$$j := 0..n \quad x_j := a + j \cdot \Delta x \quad y_j := f(x_j) \quad z_j := fp(x_j)$$

$$Z_{0,j} := x_j \quad Z_{1,j} := y_j \quad Z_{2,j} := z_j$$

	0	1	2	3	4	5	6	7	8	
Z =	0	-2	-1.6	-1.2	-0.8	-0.4	0	0.4	0.8	1.2
	1	-0.271	-0.323	-0.361	-0.359	-0.268	0	0.597	1.78	3.984
	2	-0.135	-0.121	-0.06	0.09	0.402	1	2.089	4.006	7.304



In the following example we attempt to repeat the last exercise for a multiply-defined function:

$$f(x) := \text{if}(x < 0, x - 1, \text{if}(x < 5, x^2 - 1, 1 - x^2)) \quad fp(x) := \frac{d}{dx} f(x)$$

$$a := -4 \quad b := 8 \quad n := 12 \quad \Delta x := \frac{(b - a)}{n} \quad \Delta x = 1 \quad j := 0..n \quad x_j := a + j \cdot \Delta x$$

$$y_j := f(x_j) \quad \boxed{r_j := fp(x_j)} \quad z_{0,j} := x_j \quad z_{1,j} := y_j \quad z_{2,j} := r_j$$

This calculation does not converge to a solution.

Notice, however, that evaluating $fp(x) = df/dx$ in the range selected fails to converge. You can check that the problem occurs at $fp(x_9)$, i.e., at $x_9 = 5$. The problem is that the derivative of this function is doubly-defined at that point. Thus, with the range of values of x as defined above we cannot produce a table or a plot of the function. To avoid this problem we could re-define the range by adding, say 0.1 , to a and b , i.e.,

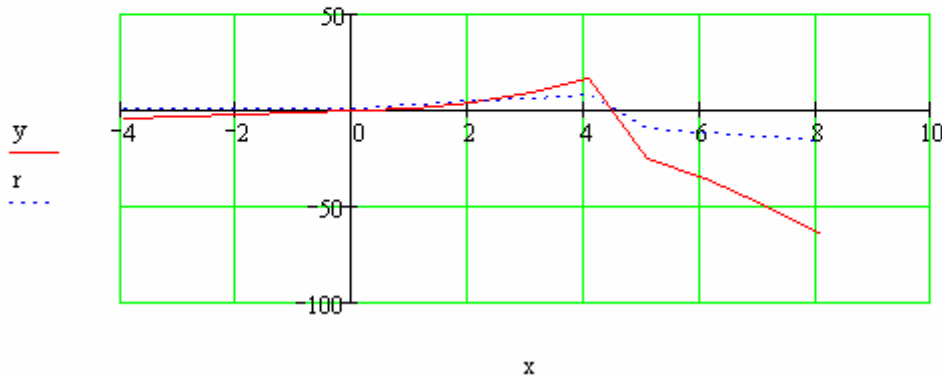
$$f(x) := \text{if}(x < 0, x - 1, \text{if}(x < 5, x^2 - 1, 1 - x^2)) \quad fp(x) := \frac{d}{dx} f(x)$$

$$a := -3.9 \quad b := 8.1 \quad n := 12 \quad \Delta x := \frac{(b - a)}{n} \quad \Delta x = 1 \quad j := 0..n \quad x_j := a + j \cdot \Delta x$$

$$y_j := f(x_j) \quad r_j := fp(x_j) \quad z_{0,j} := x_j \quad z_{1,j} := y_j \quad z_{2,j} := r_j$$

The table of values for both $f(x)$ and $fp(x)$, as well as a plot of the two functions, is shown below:

	0	1	2	3	4	5	6	7	8	9	
$z =$	0	-3.9	-2.9	-1.9	-0.9	0.1	1.1	2.1	3.1	4.1	5.1
	1	-4.9	-3.9	-2.9	-1.9	-0.99	0.21	3.41	8.61	15.81	-25.01
	2	1	1	1	1	0.2	2.2	4.2	6.2	8.2	-10.2



Symbolic evaluation of derivatives in user-defined functions

If the user-defined function uses a single expression, the symbolic evaluation is straightforward (recall that we use *Ctrl* . [*control period*] to produce the *symbolic evaluation symbol* \rightarrow):

$$g(x) := x^2 - 1 \quad \frac{d}{dx} g(x) \rightarrow 2 \cdot x$$

Symbolic evaluation of a multiply-defined function (using *Mathcad* function *if*) is not allowed, e.g.,

$$f(x) := \text{if}(x < 0, x - 1, \text{if}(x < 5, x^2 - 1, 1 - x^2))$$

$\frac{d}{dx} f(x) \rightarrow$

This value must be a scalar.

One possibility is to apply the symbolic evaluation operator to each expression in the multiply defined function as illustrated below:

$$\frac{d}{dx}(x-1) \rightarrow 1 \quad \text{for } x < 0$$

$$\frac{d}{dx}(x^2-1) \rightarrow 2 \cdot x \quad \text{for } 0 < x < 5$$

$$\frac{d}{dx}(1-x^2) \rightarrow (-2) \cdot x \quad \text{for } x > 5$$

Thus, the derivative function could be defined as follows:

$$fp(x) := \text{if}(x < 0, 1, \text{if}(x < 5, 2 \cdot x, -2 \cdot x))$$

Functions defined by summations and integrals

Consider the function $g(x)$ defined by a single expression. The following exercise shows two other functions $gs(x)$ and $gi(x)$ defined in terms of a summation and an integral of function $g(x)$. This case is straightforward. As shown below, the symbolic expressions for functions $gs(x)$ and $gi(x)$ can be easily calculated using the *symbolic evaluation symbol* \rightarrow :

$$g(x) := x^2 - 1$$

$$gs(x) := \sum_{k=1}^x g(k) \quad \sum_{k=1}^x g(k) \rightarrow \frac{1}{3} \cdot (x+1)^3 - \frac{1}{2} \cdot (x+1)^2 - \frac{5}{6} \cdot x + \frac{1}{6}$$

$$gi(x) := \int_0^x g(x) dt \quad \int_0^x g(x) dt \rightarrow x^3 - x \quad +$$

An attempt to define a function with the integral of a multiply-defined function fails:

$$f(x) := \text{if}(x < 0, x-1, \text{if}(x < 5, x^2-1, 1-x^2)) \quad \int_0^x f(s) ds \rightarrow$$

As we did in the case of the derivative of a multiply-defined function, we can define the symbolic integrals separately by range, i.e.,

$$\int_0^x (s - 1) ds \rightarrow (-x) + \frac{1}{2} \cdot x^2 \quad \text{for } x < 0$$

$$\int_0^x (s^2 - 1) ds \rightarrow \frac{1}{3} \cdot x^3 - x \quad \text{for } 0 < x < 5$$

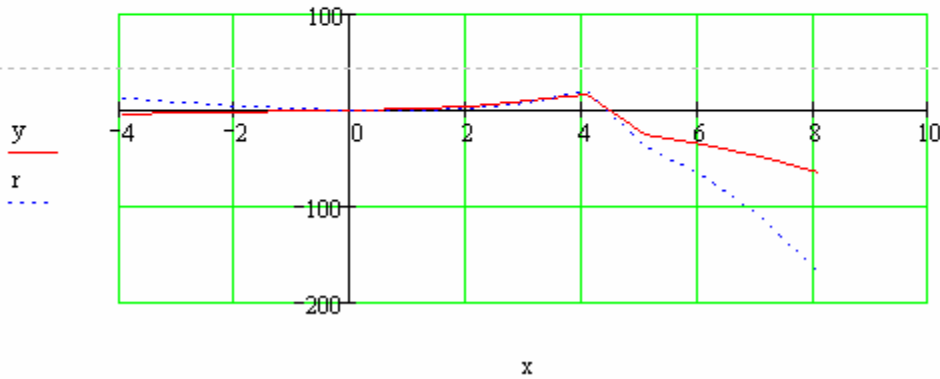
$$\int_0^x (1 - s^2) ds \rightarrow \frac{-1}{3} \cdot x^3 + x \quad \text{for } x > 5$$

Then, we can put together the integrated function, $fs(x)$, as follows:

$$fs(x) := \text{if}\left[x < 0, (-x) + \frac{1}{2} \cdot x^2, \text{if}\left(x < 5, \frac{1}{3} \cdot x^3 - x, \frac{-1}{3} \cdot x^3 + x\right)\right]$$

A table and graph of the two functions, $f(x)$ and $fs(x)$, are presented below:

	0	1	2	3	4	5	6	7
Z = 0	-3.9	-2.9	-1.9	-0.9	0.1	1.1	2.1	3.1
1	-4.9	-3.9	-2.9	-1.9	-0.99	0.21	3.41	8.61
2	11.505	7.105	3.705	1.305	-0.1	-0.656	0.987	6.83



The same result can be obtained by defining the integrated function $fs(x)$ as follows:

$$f(x) := \text{if}\left(x < 0, x - 1, \text{if}\left(x < 5, x^2 - 1, 1 - x^2\right)\right) \quad fs(x) := \int_0^x f(s) ds$$

Functions defined by a *Given-Find* block

This approach is useful if you need to calculate an implicit value in an equation. A typical case is the *Coolebrook-White* equation used to calculate the friction factor in a pipeline. The equation is given by

$$\frac{1}{\sqrt{f}} = -2 \cdot \log\left(\frac{e}{3.7 \cdot D} + \frac{2.52}{Re \cdot \sqrt{f}}\right)$$

Where f is the friction factor, Re is the Reynolds number, e is the absolute roughness of the pipe, and D is the pipe's diameter. Due to the complex nature of the equation, it is not possible to solve for f explicitly. A *Given-Find* block can be used to solve for f if the values of Re , e , and D are known, as illustrated below:

```

e := 0.0001      Re := 1·107      D := 0.5
Assuming:      f := 0.02

Given
  1/√f = -2·log(e/(3.7·D) + 2.52/(Re·√f))

f := Find(f)      f = 0.014

```

It is possible to define a function $ffact(e,D,Re)$ as follows [$ffact$ stands for *friction factor*]:

```

Initial guess for f:  f := 0.02  Given
  1/√f = -2·log(e/(3.7·D) + 2.52/(Re·√f))  ffact(e,D,Re) := Find(f)

```

A couple of evaluations of the function are shown below:

$$ffact(0.001, 0.25, 1 \times 10^6) = 0.029 \qquad ffact(0.000001, 1.5, 1 \cdot 10^5) = 0.018$$

We can use function $ffact$ to calculate the velocity of flow V in a pipe of length L , diameter D , absolute roughness e , if a fluid with kinematic viscosity ν produces a head loss h_f . The equation to use is the Darcy-Weisbach equation defined as

$$h_f = f \cdot \frac{L}{D} \cdot \frac{V^2}{2g},$$

where $f = \text{ffact}(e/D, Re)$, and $Re = VD/\nu$. To set up the equation for solution in *Mathcad* we can write the Darcy-Weisbach equation as:

$$h_f = \text{ffact}\left(\frac{e}{D}, \frac{V \cdot D}{\nu}\right) \cdot \frac{L}{D} \cdot \frac{V^2}{2g}$$

A solution for the velocity is shown next:

$$\begin{aligned} \nu &:= 1 \times 10^{-6} \frac{\text{ft}^2}{\text{s}} & e &:= 0.0001 \cdot \text{ft} & D &:= 0.5 \cdot \text{ft} & L &:= 100 \cdot \text{ft} & h_f &:= 10 \cdot \text{ft} \\ \text{Assume: } \underline{V} &:= 2.3 \cdot \frac{\text{ft}}{\text{sec}} \\ \text{Given} \\ h_f &= \text{ffact}\left(\frac{e}{D}, \frac{V \cdot D}{\nu}\right) \cdot \frac{L}{D} \cdot \frac{V^2}{2g} \\ \underline{V} &:= \text{Find}(V) & V &= 4.642 \frac{\text{m}}{\text{s}} \end{aligned}$$

The Darcy-Weisbach equation, written in terms of the discharge Q , is

$$h_f = \frac{8fLQ^2}{\pi^2 g D^5} = \text{ffact}\left(\frac{e}{D}, \frac{4Q}{\pi \nu D}\right) \cdot \frac{8LQ^2}{\pi^2 g D^5},$$

Where $Re = 4Q/(\pi \nu D)$, and $Q = (\pi D^2/4)V$. Solving the example given above in terms of the discharge produces the following *Mathcad* result:

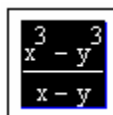
$$\begin{aligned} \underline{\nu} &:= 1 \times 10^{-6} \frac{\text{ft}^2}{\text{s}} & e &:= 0.0001 \cdot \text{ft} & D &:= 0.5 \cdot \text{ft} & L &:= 100 \cdot \text{ft} & h_f &:= 10 \cdot \text{ft} \\ \text{Assume: } Q &:= 10 \cdot \frac{\text{ft}^3}{\text{sec}} \\ \text{Given} \\ h_f &= \text{ffact}\left(\frac{e}{D}, \frac{4Q}{\pi \cdot \nu \cdot D}\right) \cdot \frac{8 \cdot L \cdot Q^2}{\pi^2 \cdot g \cdot D^5} \\ \underline{Q} &:= \text{Find}(Q) & Q &= 0.085 \frac{\text{m}^3}{\text{s}} & \text{and} & \underline{V} &:= \frac{4Q}{\pi D^2} & V &= 4.642 \frac{\text{m}}{\text{s}} \end{aligned}$$

Symbolic manipulation of algebraic expressions

The menu *Symbolics* in *Mathcad* contains functions for manipulation of algebraic expressions, e.g., *Simplify*, *Expand*, *Factor*, and *Collect*. To illustrate this point, type in your worksheet the expression:

$$\frac{x^3 - y^3}{x - y}$$

Then, click on the expression and drag the mouse through it highlighting the entire expression. The expression should look like this:



Select *Symbolics*>*Simplify* while the expression is highlighted. The result is a new field in the worksheet with the simplified expression, i.e.,

$$\frac{x^3 - y^3}{x - y}$$
$$x^2 + y \cdot x + y^2$$

Using text (“), and dragging fields around the worksheet, you can document and manipulate the operation once it is performed to make it easier to follow, e.g.,

The fraction $\frac{x^3 - y^3}{x - y}$ simplifies to $x^2 + y \cdot x + y^2$ using *Symbolics*>*Simplify*

Here are examples of the functions *Expand*, and *Factor* after manipulation (the operation of these two functions is similar to that of *Simplify*, presented above):

The square $(x - y)^2$ expands to $x^2 - 2 \cdot y \cdot x + y^2$ using *Symbolics*>*Expand*

The expression $x^2 y - 2 \cdot x \cdot y^2$ can be re-written as $x \cdot y \cdot [(-2) \cdot y + x]$ using *Symbolics*>*Factor*

+

The commands *Simplify*, *Expand*, and *Factor* can be activated using the *Symbolic Keyword Evaluation* symbol \Rightarrow , available in the *Evaluation Toolbar* $\frac{x}{=}$. This symbol can also be obtained by using *Ctrl Shift .* (control shift period). Activating the *Symbolic Keyword Evaluation* symbol produces the following field:



This placeholder is empty.

The first placeholder (to the left) is where the expression to be operated upon is written, while the second placeholder is for the user to type the operation to be performed (i.e., *simplify*, *expand*, or *factor*). The preferred filling of the placeholders is from the right to the left, thus, first type the operation name, e.g., *expand*, and then type the expression to be expanded, e.g., $(x-y)^2$.

Repeating the exercises shown above with the *Symbolic Keyword Evaluation* symbol, we get the following results:

$$\begin{aligned}\frac{x^3 - y^3}{x - y} \text{ simplify} &\rightarrow x^2 + y \cdot x + y^2 \\ (x - y)^2 \text{ expand} &\rightarrow x^2 - 2 \cdot y \cdot x + y^2 \\ x^2 \cdot y - 2 \cdot x \cdot y^2 \text{ factor} &\rightarrow x \cdot y \cdot [(-2) \cdot y + x]\end{aligned}$$

The command *collect* can be used with the *Symbolic Keyword Evaluation* symbol, by typing *collect* followed by a comma and the factor used for collecting terms. For example, in the following result we used *collect, x²* to collect all terms with x^2 in the expression shown:

$$x^2 + x^2 \cdot y + x^2 \cdot z + x \cdot z \text{ collect, } x^2 \rightarrow (1 + y + z) \cdot x^2 + x \cdot z$$

To produce a vector of polynomial coefficients, use the *Symbolic Keyword Evaluation* symbol, typing *coeffs* followed by a comma and the polynomial variable. The following examples show the coefficients of four third-order polynomials:

$$x^3 \text{ coeffs, } x \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad x^3 + 3x^2 \text{ coeffs, } x \rightarrow \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

$$x^3 + 3x^2 - 2x \text{ coeffs, } x \rightarrow \begin{pmatrix} 0 \\ -2 \\ 3 \\ 1 \end{pmatrix} \qquad x^3 + 3x^2 - 2x + 2 \text{ coeffs, } x \rightarrow \begin{pmatrix} 2 \\ -2 \\ 3 \\ 1 \end{pmatrix}$$

Notice that the vector of coefficients is filled vertically downwards from the lowest to the highest powers of the polynomial variable.

Roots of a polynomial

The following application illustrates the use of the polynomial coefficients operation in the calculation of the roots of a polynomial. First, the polynomial is defined as $p(x)$. Then, the polynomial coefficients are stored in variable v by using the *Symbolic Keyword Evaluation* symbol \rightarrow and *coeffs*. Finally, function *polyroots* is used to calculate the roots of $p(x)$:

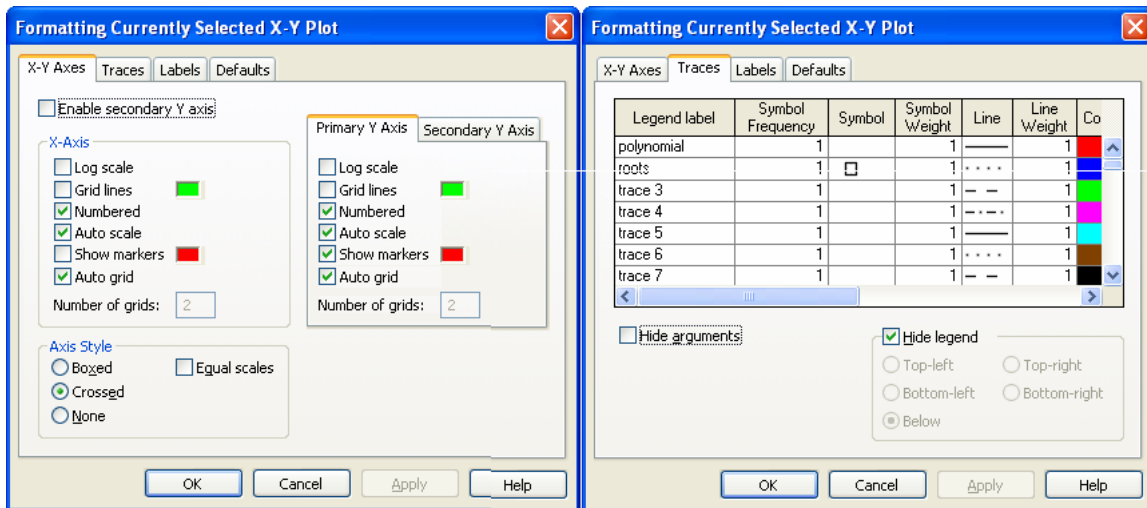
$$p(x) := x^4 - 2 \cdot x^3 + x^2 - 5 \cdot x + 2 \qquad v := p(x) \text{ coeffs}, x \rightarrow \begin{pmatrix} 2 \\ -5 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

$$r := \text{polyroots}(v) \qquad r = \begin{pmatrix} -0.372 + 1.394i \\ -0.372 - 1.394i \\ 0.412 \\ 2.333 \end{pmatrix}$$

Notice that two of the roots of the polynomial shown above are complex numbers, namely, $-0.372+1.394i$ and $-0.372-1.394i$. The polynomial has two real roots, 0.412 and 2.333 .

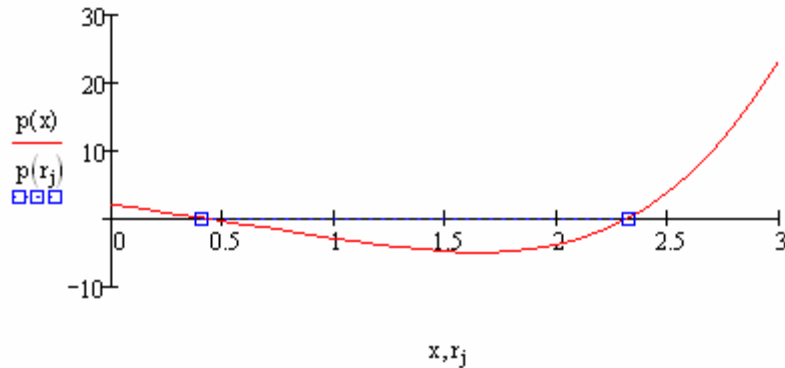
Visualizing the roots of the polynomial

To visualize the real roots, we can produce a plot of the polynomial. In the following example, we produce a range for x , namely, $x:=0,0.1...3$ (i.e., $x := 0, 0.1;3$), and a range for $j:=1,2,...,4$ for the indices of r . The plot is produced with the abscissa placeholder containing x and r_j , and the ordinate placeholder containing $f(x)$ and $f(r_j)$. To highlight the roots, make sure that, after the plot is produced, the *Format* tabs are as shown below:



The results are shown in the next page:

$x := 0, 0.1 \dots 3$ $j := 1, 2 \dots 4$



Symbolic solution to single equations

In this section we present two methods for the symbolic solution of single equations. The first method consists of using the *Symbolic Keyword Evaluation* symbol (*control shift period*) with the keyword *solve*. As an exercise, try the solution of the polynomial $x^3 + 10x^2 + 19x - 30 = 0$, as follows:

1. Type *Cntl Shift* . (*control shift period*)
2. In the second placeholder type *solve, x*
3. In the first placeholder type the polynomial $x^3 + 10x^2 + 19x - 30$ *Cntl* = 0
4. Click outside of the expression

The result is the following:

$$x^3 + 10 \cdot x^2 + 19 \cdot x - 30 = 0 \text{ solve, } x \rightarrow \begin{pmatrix} -5 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ -6 \end{pmatrix}$$

This result indicates that the polynomial has the roots $x = -5$, $x = 1$, and $x = -6$.

The second method of symbolic solution uses a *Given-Find* block as illustrated below:

Given

$$x^3 + 10 \cdot x^2 + 19 \cdot x - 30 = 0$$

$$\text{Find}(x) \rightarrow (-5 \ 1 \ -6)$$

The arrow after the *Find* particle is produced by typing *Cntl* . (*control period*).

Symbolic solutions need not produce numbers as results. As an example consider the equation $v^2 = v_0^2 + 2as$. Suppose that we want to solve for the variable a . In such case, we can use one of the following approaches:

1 – Using *solve*:

$$v^2 = v_0^2 + 2 \cdot a \cdot s \text{ solve, } a \rightarrow \frac{-1}{2} \cdot \frac{(-v^2) + v_0^2}{s}$$

2 – Using a *Given-Find* block:

$$\begin{array}{l} \text{Given} \\ v^2 = v_0^2 + 2 \cdot a \cdot s \\ \\ \text{Find}(a) \rightarrow \frac{-1}{2} \cdot \frac{(-v^2) + v_0^2}{s} \end{array}$$

Symbolic solution to systems of equations

The two approaches shown above for symbolic solutions of single equations can be used for solving systems of equations, if the equations and unknowns are grouped as column vectors. Consider the following system of equations whose solutions are numbers:

$$\begin{pmatrix} x^2 + y^2 = 13 \\ x \cdot y = 6 \end{pmatrix} \text{ solve, } \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ -2 & -3 \\ -3 & -2 \end{pmatrix}$$

To write this expression follow the steps shown in the first example of the previous section, except that you will need to type arrays before and after the *solve* particle. To enter an array, type *Cntl-M* and select, for this case, 2 rows and 1 column. Then replace the placeholders with equations or unknowns.

A solution of the same system of equations using a *Given-Find* block is shown next:

$$\begin{array}{l} \text{Given} \\ x^2 + y^2 = 13 \\ \\ x \cdot y = 6 \\ \\ \text{Find}(x,y) \rightarrow \begin{pmatrix} 3 & 2 & -2 & -3 \\ 2 & 3 & -3 & -2 \end{pmatrix} \end{array}$$

The solutions shown above correspond to the pairs $(x,y) = (3,2), (2,3), (-2,-3), (-3,-2)$.

Symbolic solutions of systems of equations need not be numbers, as illustrated in the following example. The equations listed to the left are those for position and velocity in a uniformly-accelerated motion. The expression below uses *Cntl Shift* . (*control shift period*) and *solve* to solve simultaneously for v_0 and a :

$$\left(\begin{array}{l} x = x_0 + v_0 \cdot t - \frac{1}{2} \cdot a \cdot t^2 \\ v = v_0 + a \cdot t \end{array} \right) \text{solve,} \left(\begin{array}{l} v_0 \\ a \end{array} \right) \rightarrow \left(\begin{array}{l} \frac{1}{3} \cdot \frac{t \cdot v + 2 \cdot x - 2 \cdot x_0}{t} \\ \frac{-2}{3} \cdot \frac{x - x_0 - t \cdot v}{t^2} \end{array} \right)$$

The solutions to the system are: $v_0 = \frac{vt + 2(x - x_0)}{3t}$, $a = -\frac{2}{3t^2}(x - x_0 - vt)$.

An alternative solution is shown next using a *Given-Find* block:

Given

$$x = x_0 + v_0 \cdot t - \frac{1}{2} \cdot a \cdot t^2$$

$$v = v_0 + a \cdot t$$

$$\text{Find}(v_0, a) \rightarrow \left(\begin{array}{l} \frac{1}{3} \cdot \frac{t \cdot v + 2 \cdot x - 2 \cdot x_0}{t} \\ \frac{-2}{3} \cdot \frac{x - x_0 - t \cdot v}{t^2} \end{array} \right)$$

The following example shows the solution to a system of linear equations. Since the solution is symbolic, the results are given as integers or fractions:

$$\left(\begin{array}{l} 2 \cdot x - 3 \cdot y + 4 \cdot z = 5 \\ 5x - 2y + z = 3 \\ x + y + z = 1 \end{array} \right) \text{solve,} \left(\begin{array}{l} x \\ y \\ z \end{array} \right) \rightarrow \left(\begin{array}{l} \frac{11}{34} \\ \frac{-4}{17} \\ \frac{31}{34} \end{array} \right)$$

Given-Find blocks can be used to produce symbolic (\rightarrow) as well as numeric results ($=$) for systems of linear equations. The first example shows a symbolic result.

Given

$$2 \cdot x - 3 \cdot y + 4 \cdot z = 5$$

$$5x - 2y + z = 3$$

$$x + y + z = 1$$

$$\text{Find}(x, y, z) \rightarrow \left(\begin{array}{l} \frac{11}{34} \\ \frac{-4}{17} \\ \frac{31}{34} \end{array} \right)$$

A numerical result requires a first guess for the unknowns:

$$\begin{aligned}
 x &:= 6 & y &:= 4 & z &:= 3 \\
 \text{Given} & & & & & \\
 2 \cdot x - 3 \cdot y + 4 \cdot z &= 5 \\
 5x - 2y + z &= 3 \\
 x + y + z &= 1 \\
 \text{Find}(x, y, z) &= \begin{pmatrix} 0.324 \\ -0.235 \\ 0.912 \end{pmatrix}
 \end{aligned}$$

Working with complex numbers

To enter the unit imaginary number while typing a complex number, use $1i$ instead of simply i , e.g., type the following complex numbers $z1:=3-2*1i$ and $z2:=5-3*1i$. The result is the following values:

$$z1 := 3 - 2i \qquad z2 := 5 - 3i$$

Some basic arithmetic operations using these complex numbers are shown below:

$$\begin{aligned}
 z1 &:= 3 - 2i & z2 &:= 5 - 3i \\
 z1 + z2 &= 8 - 5i & z1 - z2 &= -2 + i \\
 z1 \cdot z2 &= 9 - 19i & \frac{z1}{z2} &= 0.618 - 0.029i \\
 \frac{1}{z1} &= 0.231 + 0.154i & z2^2 &= 16 - 30i
 \end{aligned}$$

Trigonometric, exponential, and logarithmic functions can be applied to complex numbers, e.g.,

$$\begin{aligned}
 \sin(z1) &= 0.531 + 3.591i & \arccos(z2) &= 0.547 + 2.453i \\
 \exp(z1) &= -8.359 - 18.264i & \ln(z2) &= 1.763 - 0.54i
 \end{aligned}$$

The absolute value function (use $|$) provides the modulus of a complex number, whereas function arg provides the argument. Functions Re and Im represent the *real* and *imaginary* parts of a complex number. To calculate a *complex conjugate*, type the complex variable name (or complex number), and press “ $\bar{}$ ”. Some examples are shown below:

$$\begin{aligned}
 |z1| &= 3.606 & \arg(z1) &= -0.588 \\
 \text{Re}(z1) &= 3 & \text{Im}(z1) &= -2 \\
 \overline{z1} &= 3 + 2i & \overline{z2} &= 5 + 3i
 \end{aligned}$$

A complex number of modulus r and argument θ , can be written as $z = re^{i\theta}$. This is referred to as the *polar representation* of the complex number, as opposite to the *Cartesian representation*, $z = x+iy$. The following example shows the definition of a complex number in polar representation and its equivalent Cartesian representation:

$$z3 := 2.5 \cdot e^{\frac{-i\pi}{6}} \quad z3 = 2.165 - 1.25i$$

Complex solutions to equations

If an equation has a complex solution, it is possible to use a *Given-Find* block, with a complex initial guess, to find that solution. This is illustrated with the following examples:

$$x := 1 + i$$

Given

$$x^2 - 4x + 13 = 0$$

$$x := \text{Find}(x) \quad x = 2 - 3i$$

Graphics of matrices (example using complex functions)

A complex variable is the variable $z = x+iy = re^{i\theta}$. A function of a complex variable $F(z)$ is also a complex variable, and so it can be written as

$$F(z) = \phi(x,y) + i\psi(x,y) = \phi(r, \theta) + i\psi(r, \theta)$$

The following exercise shows how to plot the real and imaginary part of the complex function $F(z) = z^{1/2}$, by creating a grid of values in x and y . The values of x and y generated below are vectors, which are used to put together a matrix of values

$$z_{i,j} = x_i + iy_j.$$

The values of the complex function are stored in $\Phi_{i,j} = F(z_{i,j})$. The values of $\phi_{i,j}$ and $\psi_{i,j}$ are matrices containing the real and imaginary parts of $\Phi_{i,j}$.




$$F(z) := z^{\frac{1}{2}}$$

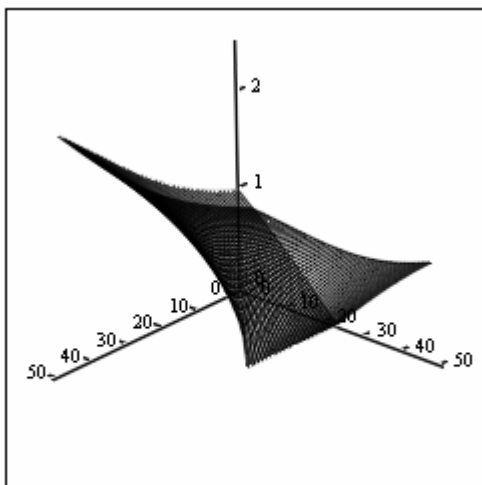
$$N := 50 \quad M := N \quad X_{\min} := -5. \quad X_{\max} := 5. \quad Y_{\min} := X_{\min} \quad Y_{\max} := X_{\max}$$

$$\Delta x := \frac{X_{\max} - X_{\min}}{N} \quad \Delta x = 0.2 \quad \Delta y := \frac{Y_{\max} - Y_{\min}}{M} \quad \Delta y = 0.2$$

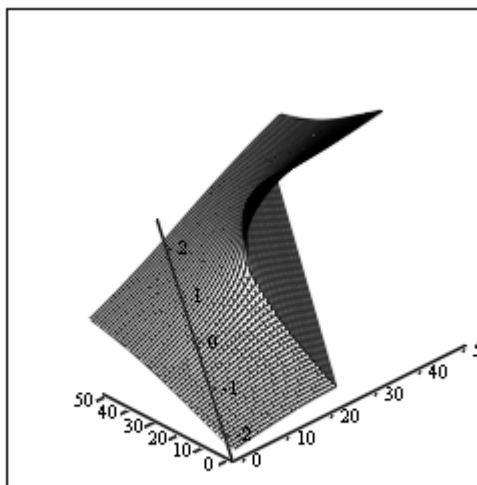
$$i := 0..N \quad x_i := X_{\min} + i \cdot \Delta x \quad j := 0..M \quad y_j := Y_{\min} + j \cdot \Delta y$$

$$z_{i,j} := x_i + y_j \cdot i \quad \Phi_{i,j} := F(z_{i,j}) \quad \phi_{i,j} := \text{Re}(\Phi_{i,j}) \quad \psi_{i,j} := \text{Im}(\Phi_{i,j})$$

To visualize these values we can use either a surface plot , or a contour plot . These plots are available from the *Graph Toolbar* which can be activated by pressing the *Graph Toolbar* button in the *Mathcad* toolbar . These two graphs have a single placeholder. Type the name of the matrix to be plotted in that placeholder. The figures below show the real part, ϕ , and the imaginary part, ψ , of the complex function $F(z)$ in the form of surface plots.

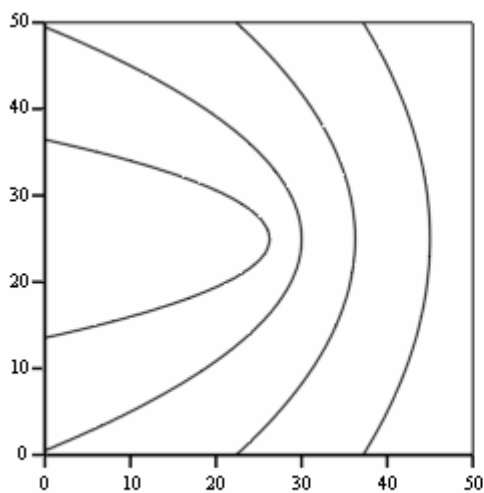


ϕ

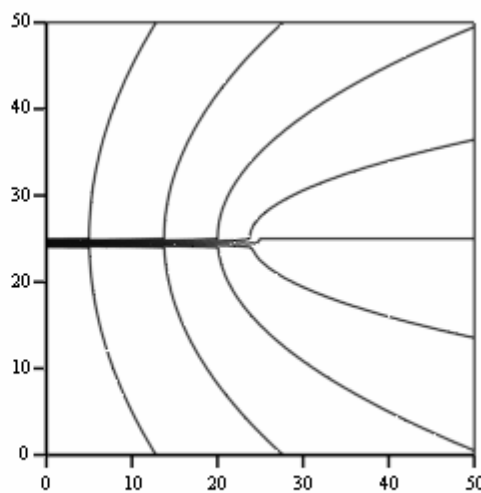


ψ

The same data is plotted next using contour plots:



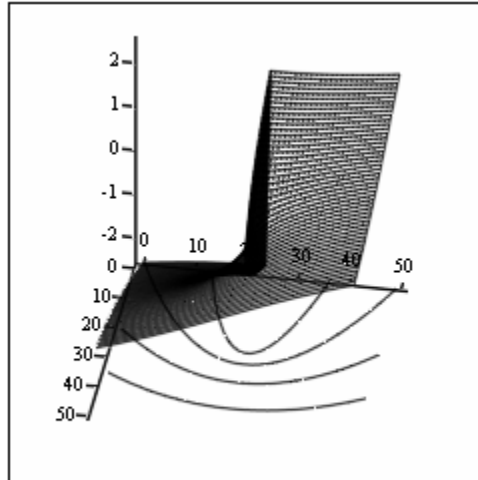
ϕ



ψ

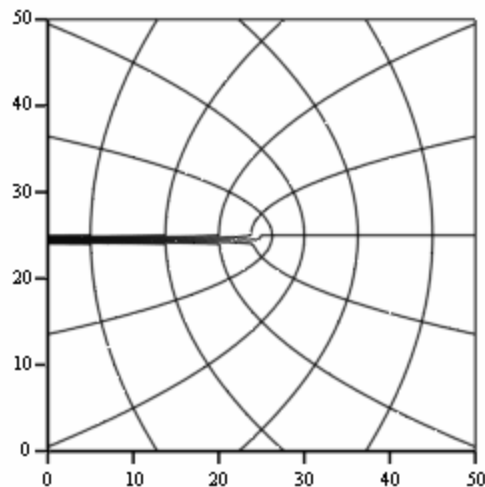
Visualization of potential flows using complex functions

In the description of potential flow, a function such as $F(z)$ is referred to as a *complex potential* (or *complex velocity potential*), $\phi(x,y)$ is the *velocity potential*, and $-\psi(x,y)$. The contours of $\phi(x,y)$ represent the *equipotential lines* of the flow, while the contours of $\psi(x,y)$ represent the *streamlines* of the flow. The combination of these two plots represents the *flow net* of this flow. To produce a flow net, produce a contour plot and type ϕ, ψ in the placeholder. The first version of the plot would actually show a contour plot combined with a surface plot, e.g.,




ϕ, ψ

To produce a double contour plot, click on the plot, do a right-click and select the *Properties* option. In the resulting form, click on the *Plot 2* tab, and select *Contour Plot*, then press the [OK] button. The resulting graph is the flow net of the flow corresponding to the complex potential $F(z) = z^{1/2}$.

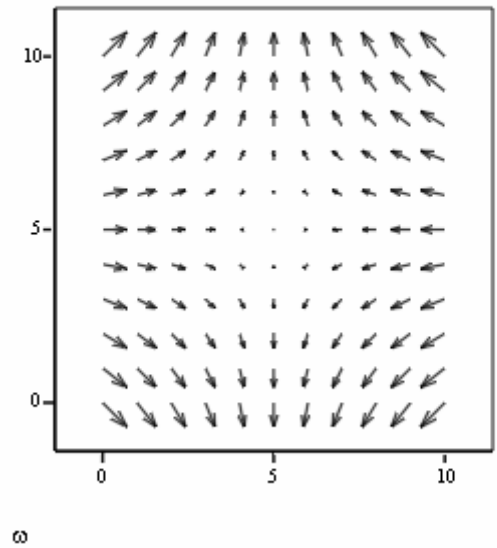


ϕ, ψ

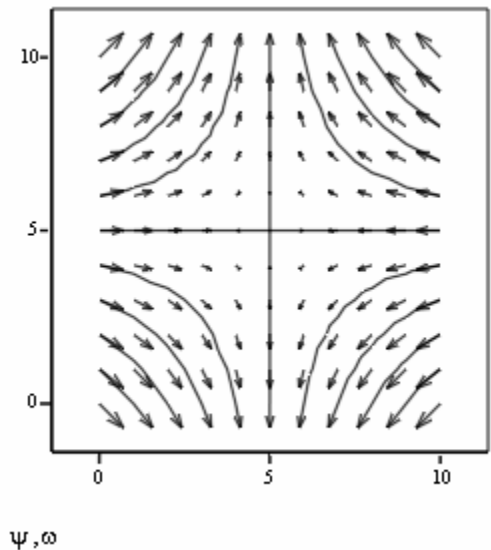
Vector field plot for flow velocities from a complex function

The function $w = dF/dz$ is referred to as the *complex velocity*, and is such that $w = -u+iv$, where u and v are the velocity components of the flow. The following commands use the *Vector Field Plot* button () in the *Graph Toolbar*. The grid for the vector field was generated in a similar fashion to that used in the previous exercise, but reducing the values of M and N to 10. The complex velocity matrix is $\omega_{i,j}$, and it's calculated using the negative complex conjugate of $w(z)$. The function used for this case was $F(x):=z^2$.



$$w(z) := \frac{d}{dz}F(z) \quad \omega_{i,j} := w(z_{i,j}) \quad \overline{\omega_{i,j}} := -\omega_{i,j}$$



A combined plot consisting of the contour plot of $f(x,y)$ superimposed to the velocity field is shown next. Notice that the streamlines are tangential to the velocity vectors:



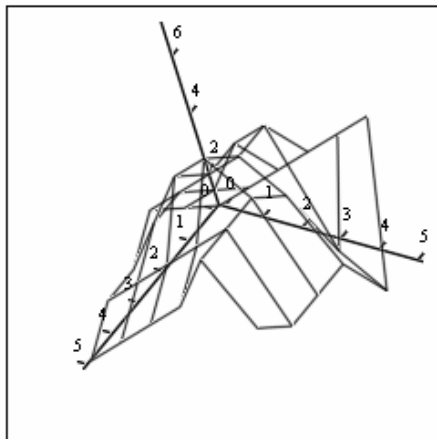
Additional matrix plots

While the examples shown above were derived from complex functions, matrices can be created by typing in values into an array. To define matrices use the *Vector and Matrix Toolbar* activated with the *Vector and Matrix Toolbar button* in the toolbar . Within this toolbar press the *Matrix or Vector button* . Select a matrix with 6 rows and six columns assigned to a variable M and enter the values shown below (or other values, if you want). Then, define a matrix $N = 2M - 3$. Use the surface plot and contour plot buttons in the *Graph Toolbar* to produce the plots shown below.

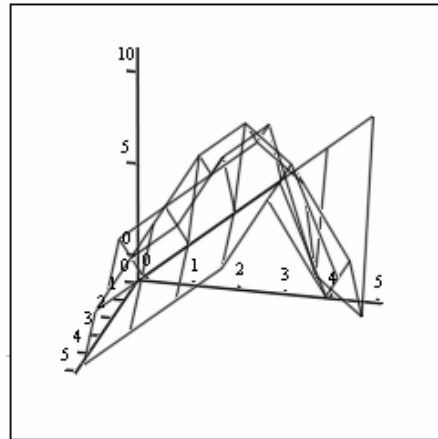
$$M := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 2 & 1 \\ 3 & 4 & 5 & 6 & 5 & 3 \\ 2 & 4 & 6 & 7 & 6 & 3 \\ 2 & 3 & 4 & 5 & 6 & 3 \\ 1 & 2 & 3 & 4 & 6 & 4 \end{pmatrix}$$

$$N := 2 \cdot M - 3$$

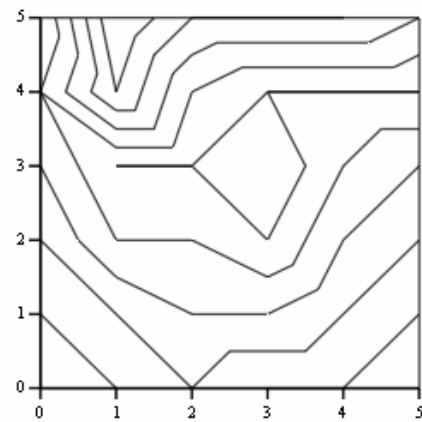
$$N = \begin{pmatrix} -1 & 1 & 3 & 5 & 7 & 9 \\ 1 & 3 & 7 & 9 & 1 & -1 \\ 3 & 5 & 7 & 9 & 7 & 3 \\ 1 & 5 & 9 & 11 & 9 & 3 \\ 1 & 3 & 5 & 7 & 9 & 3 \\ -1 & 1 & 3 & 5 & 9 & 5 \end{pmatrix}$$



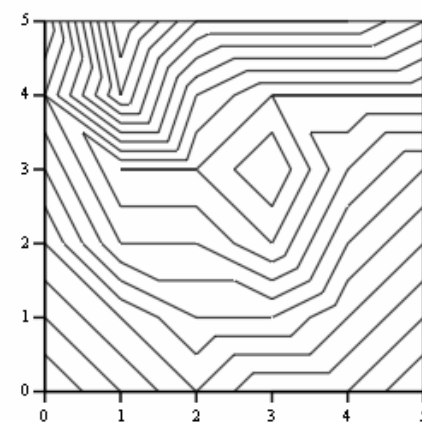
M



N



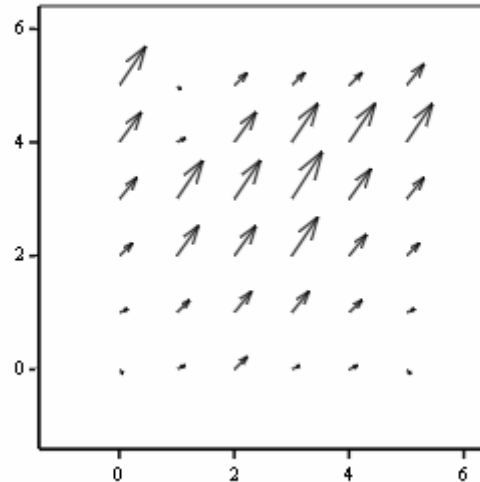
M



N

A matrix of velocity vectors can be defined by using $V = M + N \cdot i$. A plot of the velocity vectors is produced by using the *Vector Field Plot* button in the *Graph Toolbar*:

$$\underline{V} := M + N \cdot j$$



\underline{V}

A second example of a vector field plot is presented next. Velocity components $u(x,y)$ and $v(x,y)$ are given, and arrays of values of x and y are generated to populate a grid in the x - y plane. Velocity vectors $U_{i,j}$ and $V_{i,j}$ are generated as

$$U_{i,j} = u(x_i, y_j) \text{ and } V_{i,j} = v(x_i, y_j).$$

To plot the vectors it is necessary to create a complex matrix

$$W_{i,j} = U_{i,j} + i V_{i,j}.$$

Then, a *Vector Field Plot* is generated by typing W in the graph placeholder:

$$u(x,y) := x + y$$

$$v(x,y) := x - y$$

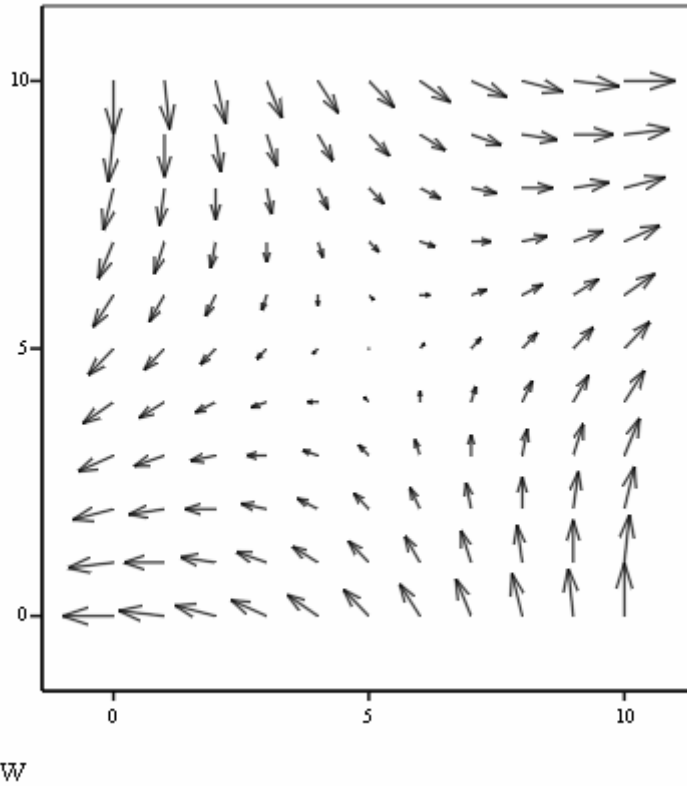
$$a := -5 \quad b := 5 \quad n := 10 \quad \Delta x := \frac{b - a}{n} \quad \Delta x = 1 \quad i := 0..n \quad x_i := a + i \cdot \Delta x$$

$$c := -5 \quad d := 5 \quad m := 10 \quad \Delta y := \frac{d - c}{m} \quad \Delta y = 1 \quad j := 0..n \quad y_j := c + j \cdot \Delta y$$

$$U_{i,j} := u(x_i, y_j)$$

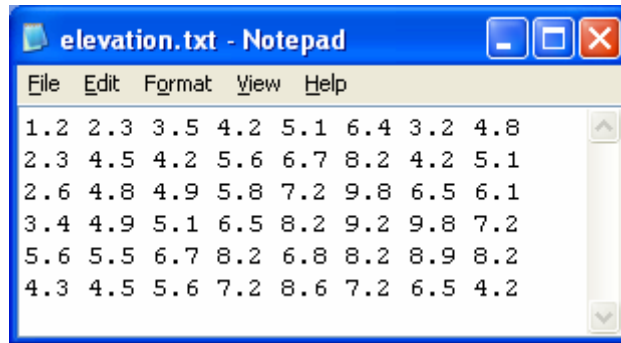
$$V_{i,j} := v(x_i, y_j)$$

$$W_{i,j} := U_{i,j} + i \cdot V_{i,j}$$

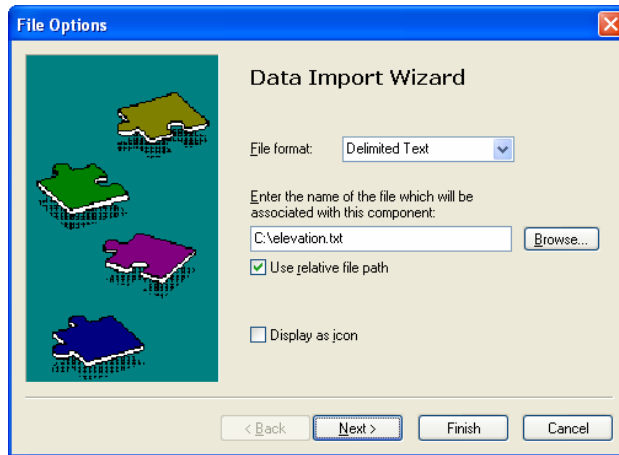


Entering data into a matrix

Suppose that we are going to bring in into *Mathcad* data from a text file. Say that the data is available as a matrix in the text file *elevation.txt*, which looks as follows:



The easiest way to enter data is to use the *Data Import Wizard* (*Insert>Data> Data Import Wizard*). The resulting input form is the following (after browsing for file *c:\elevation.txt*). Press the [Next] button three times to go through a number of options for reading the file. Then press the [Finish] button. The result is shown below.



:=

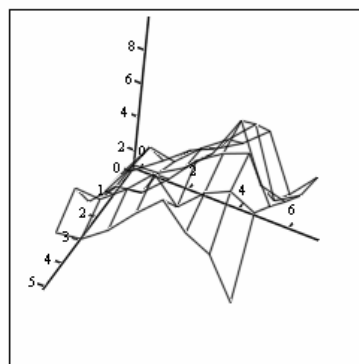
	0	1	2	3	4	5
0	1.2	2.3	3.5	4.2	5.1	6.4
1	2.3	4.5	4.2	5.6	6.7	8.2
2	2.6	4.8	4.9	5.8	7.2	9.8
3	3.4	4.9	5.1	6.5	8.2	9.2
4	5.6	5.5	6.7	8.2	6.8	8.2
5	4.3	4.5	5.6	7.2	8.6	7.2

Type M into the placeholder to make an assignment to a matrix M .

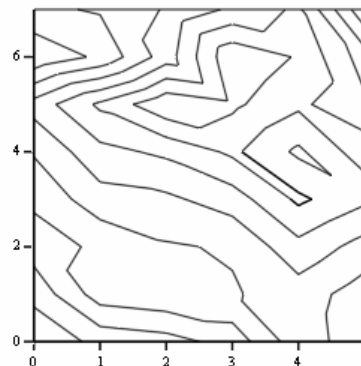
M :=

	0	1	2	3	4	5
0	1.2	2.3	3.5	4.2	5.1	6.4
1	2.3	4.5	4.2	5.6	6.7	8.2
2	2.6	4.8	4.9	5.8	7.2	9.8
3	3.4	4.9	5.1	6.5	8.2	9.2
4	5.6	5.5	6.7	8.2	6.8	8.2
5	4.3	4.5	5.6	7.2	8.6	7.2

Once entered into memory, the data can be manipulated or plotted as needed. Surface and contour plots of matrix M are shown below:



M



M

Operations with matrices

The *Vector and Matrix* toolbar contains some functions that can be used to manipulate matrices.



For example, for the matrix M , entered above from file *elevation.txt*, we can use function M^T to calculate the transpose of the matrix, or function $M^{<}$ to extract columns of the matrix:

$$M^T = \begin{pmatrix} 1.2 & 2.3 & 2.6 & 3.4 & 5.6 & 4.3 \\ 2.3 & 4.5 & 4.8 & 4.9 & 5.5 & 4.5 \\ 3.5 & 4.2 & 4.9 & 5.1 & 6.7 & 5.6 \\ 4.2 & 5.6 & 5.8 & 6.5 & 8.2 & 7.2 \\ 5.1 & 6.7 & 7.2 & 8.2 & 6.8 & 8.6 \\ 6.4 & 8.2 & 9.8 & 9.2 & 8.2 & 7.2 \\ 3.2 & 4.2 & 6.5 & 9.8 & 8.9 & 6.5 \\ 4.8 & 5.1 & 6.1 & 7.2 & 8.2 & 4.2 \end{pmatrix} \quad M^{<3>} = \begin{pmatrix} 4.2 \\ 5.6 \\ 5.8 \\ 6.5 \\ 8.2 \\ 7.2 \end{pmatrix} \quad M^{<5>} = \begin{pmatrix} 6.4 \\ 8.2 \\ 9.8 \\ 9.2 \\ 8.2 \\ 7.2 \end{pmatrix}$$

In the following exercise, we store the transpose matrix in variable MT and extract columns of matrix MT :

$$MT := M^T \quad MT^{<3>} = \begin{pmatrix} 3.4 \\ 4.9 \\ 5.1 \\ 6.5 \\ 8.2 \\ 9.2 \\ 9.8 \\ 7.2 \end{pmatrix} \quad MT^{<5>} = \begin{pmatrix} 4.3 \\ 4.5 \\ 5.6 \\ 7.2 \\ 8.6 \\ 7.2 \\ 6.5 \\ 4.2 \end{pmatrix}$$

Determinants and inverse matrices are only applicable to square matrices. In the following exercise we define a square matrix A and calculate its matrix inverse (X^{-1}) and its determinant $|X|$:

$$A := \begin{pmatrix} 1 & -2 & 4 \\ 6 & -3 & 2 \\ -8 & 9 & 1 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} -0.147 & 0.266 & 0.056 \\ -0.154 & 0.231 & 0.154 \\ 0.21 & 0.049 & 0.063 \end{pmatrix} \quad |A| = 143$$

Addition, subtraction, and multiplication of matrices are illustrated in the following examples:

$$A := \begin{pmatrix} 1 & -2 & 4 \\ 6 & -3 & 2 \\ -8 & 9 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 3 & -4 & 2 \\ 8 & -2 & -5 \\ 4 & 2 & 0 \end{pmatrix} \quad A + B = \begin{pmatrix} 4 & -6 & 6 \\ 14 & -5 & -3 \\ -4 & 11 & 1 \end{pmatrix}$$

$$A - B = \begin{pmatrix} -2 & 2 & 2 \\ -2 & -1 & 7 \\ -12 & 7 & 1 \end{pmatrix} \quad A \cdot B = \begin{pmatrix} 3 & 8 & 12 \\ 2 & -14 & 27 \\ 52 & 16 & -61 \end{pmatrix} \quad B \cdot A = \begin{pmatrix} -37 & 24 & 6 \\ 36 & -55 & 23 \\ 16 & -14 & 20 \end{pmatrix}$$

For term-by-term operations, use the *Vectorize* function ($\vec{(\cdot)}$), e.g.,

$$\vec{(A + B)} = \begin{pmatrix} 4 & -6 & 6 \\ 14 & -5 & -3 \\ -4 & 11 & 1 \end{pmatrix} \quad \vec{(A \cdot B)} = \begin{pmatrix} 3 & 8 & 8 \\ 48 & 6 & -10 \\ -32 & 18 & 0 \end{pmatrix} \quad \vec{(A - B)} = \begin{pmatrix} -2 & 2 & 2 \\ -2 & -1 & 7 \\ -12 & 7 & 1 \end{pmatrix}$$

Matrices used in the solution of systems of linear equations

Given the set of linear equations

$$\begin{aligned} x - 2y + 4z &= 22 \\ 6x - 3y + 2z &= 17 \\ -8x + 9y + z &= 42 \end{aligned}$$

it is possible to write the matrix equation $\mathbf{Ax} = \mathbf{b}$, where the matrix \mathbf{A} and the column vector \mathbf{b} are defined as follows using *Mathcad*:

$$A := \begin{pmatrix} 1 & -2 & 4 \\ 6 & -3 & 2 \\ -8 & 9 & 1 \end{pmatrix} \quad \mathbf{b} := \begin{pmatrix} 22 \\ 17 \\ 42 \end{pmatrix}$$

The solution of the matrix equation is calculated using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Using *Mathcad*:

$$x := A^{-1} \cdot b \quad x = \begin{pmatrix} 3.636 \\ 7 \\ 8.091 \end{pmatrix}$$

Thus, the solution is $x = 3.636, y = 7, z = 8.091$. To verify the solution, try:

$$xcheck := A \cdot x \quad xcheck = \begin{pmatrix} 22 \\ 17 \\ 42 \end{pmatrix}$$

Eigenvalues and eigenvectors of a square matrix

Functions $eigenvals(A)$ and $eigenvecs(A)$ are used to calculate the eigenvalues, λ , and eigenvectors, x , of a matrix A . The eigenvalue problem is described by $Ax = \lambda x$. An example of eigenvalues and eigenvectors calculation is shown below. The result from the $eigenvecs$ function is a matrix with eigenvectors in its columns.

$$A := \begin{pmatrix} 1 & -2 & 4 \\ -2 & -3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \quad eigenvals(A) = \begin{pmatrix} -5.828 \\ 5 \\ -0.172 \end{pmatrix} \quad eigenvecs(A) = \begin{pmatrix} 0.5 & 0.707 & -0.5 \\ 0.707 & 0 & 0.707 \\ -0.5 & 0.707 & 0.5 \end{pmatrix}$$

These results indicate that the eigenvalues and eigenvectors of the matrix A shown above are:

$$\begin{aligned} \lambda_1 &= -5.828, \mathbf{x}_1 = [0.5, 0.707, -0.5], \\ \lambda_2 &= 5, \mathbf{x}_2 = [0.707, 0, 0.707], \text{ and} \\ \lambda_3 &= -0.172, \mathbf{x}_3 = [-0.5, 0.707, 0.5]. \end{aligned}$$

The generalized eigenvector problem is defined by $Ax = \lambda Bx$, where A and B are square matrices of the same dimensions. *Mathcad* provides functions $genvals$ and $genvecs$ to calculate the eigenvalues and eigenvectors, respectively, of the generalized problem. Consider the following example:

$$A := \begin{pmatrix} 1 & -2 & 4 \\ -2 & -3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 3 & -1 & 5 \\ -9 & 0 & 4 \\ 2 & -1 & 4 \end{pmatrix}$$

$$genvals(A,B) = \begin{pmatrix} -0.038 \\ 0.661 \\ -15.161 \end{pmatrix} \quad genvecs(A,B) = \begin{pmatrix} -0.669 & 0.358 & 0.06 \\ 1 & 0.257 & 1 \\ 0.664 & 1 & 0.18 \end{pmatrix}$$

These results indicate that the eigenvalues and eigenvectors for the generalized problem with matrices A and B as shown above are:

$$\lambda_1 = -0.038, \mathbf{x}_1 = [-0.669, 0.358, 0.06],$$

$$\lambda_2 = 0.661, \mathbf{x}_2 = [0.358, 0.257, 1], \text{ and}$$

$$\lambda_3 = -15.161, \mathbf{x}_3 = [0.06, 1, 0.18].$$

Operations with vectors

Besides the addition, subtraction, and matrix multiplications, vector operations include the dot product ($\mathbf{u} \cdot \mathbf{v}$), and the cross product ($\mathbf{u} \times \mathbf{v}$) for three-dimensional vectors. Some vector operations are illustrated below. The absolute value operation produces the length of the vector. (Make sure that you don't use the *Determinant* ($|x|$) operation from the *Matrix and Vector* toolbar.) The operation $\sum \mathbf{v}$ represents the sum of elements of the vector.

$$\mathbf{u} := \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} \quad \mathbf{v} := \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 8 \end{pmatrix} \quad \mathbf{u} - \mathbf{v} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix} \quad |\mathbf{u}| = 6.164 \quad |\mathbf{v}| = 4.123$$

$$\mathbf{u} \cdot \mathbf{v} = 5 \quad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 16 \\ 19 \\ 2 \end{pmatrix} \quad \sum \mathbf{v} = 3 \quad \overrightarrow{(\mathbf{u} + \mathbf{v})} = \begin{pmatrix} -1 \\ 0 \\ 8 \end{pmatrix} \quad \overrightarrow{(\mathbf{u} \cdot \mathbf{v})} = \begin{pmatrix} -6 \\ -4 \\ 15 \end{pmatrix}$$

Operations with elements of matrices and vectors

Use sub-indices to extract elements of matrices and vectors. Sub-indices can be entered by using the sub-index button (\mathbf{x}_2) or by using [after the variable name. Some examples of operations on elements of matrices and vectors are shown below, using the matrices A and B , and the vectors u and v , described above:

$$A_{1,2} = 2 \quad B_{2,2} + A_{2,1} = 6 \quad u_1 + v_2 = 5 \quad B_{0,1} + A_{0,0} = 0$$

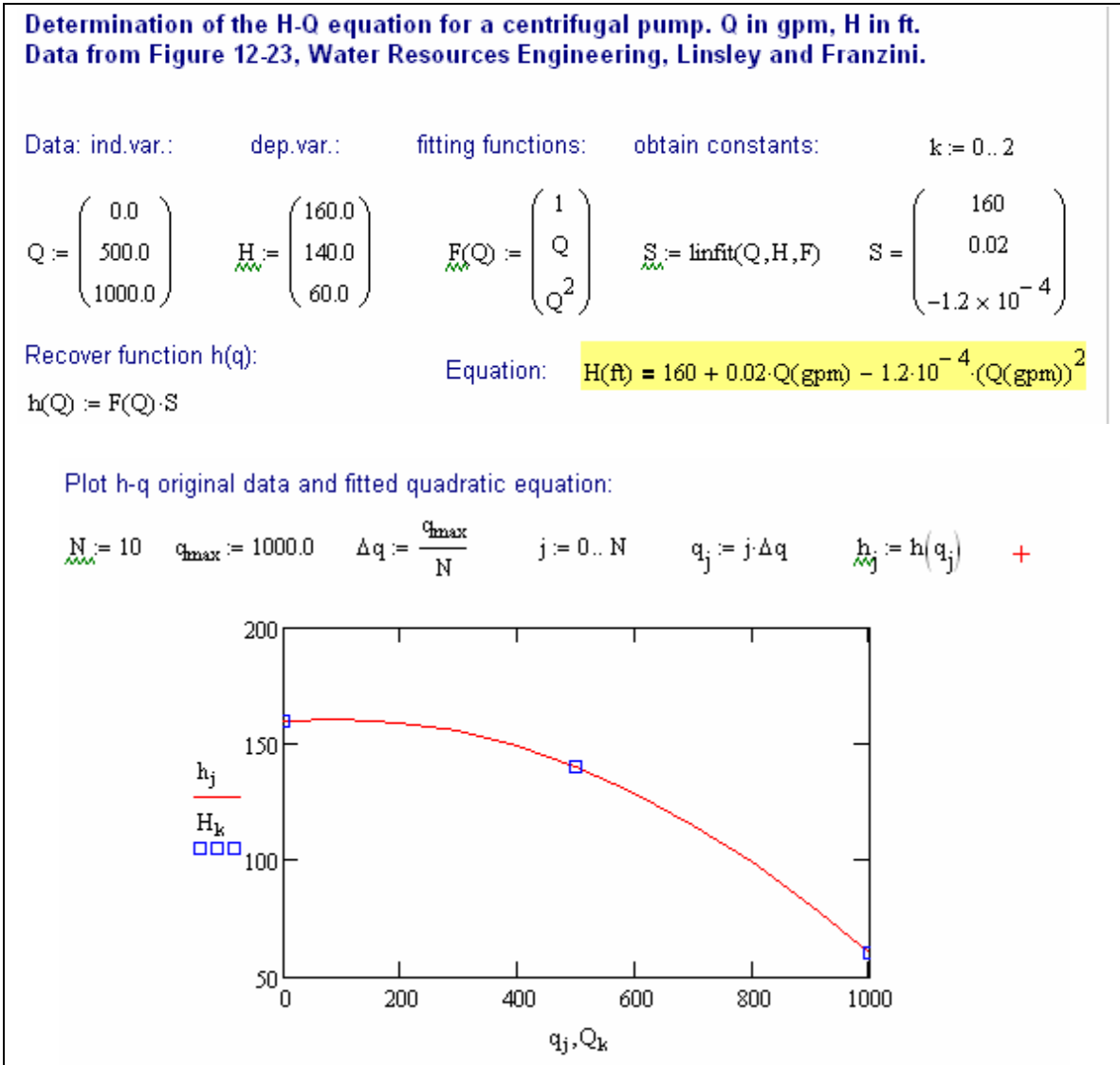
Recall that, by default, indices in *Mathcad* start at 0, rather than 1. [Note: To change the origin of array indices to 1, type *ORIGIN:=1* before using the array.]

Data fitting using vectors (*linfit*)

The following example shows the fitting of discharge (Q) and head (H) data to a quadratic equation of the form $H = a + bQ + cQ^2$, given three data points as shown in the worksheet. The values of Q and H are entered as column vectors. Function $F(Q)$ is a vector function showing the functions whose linear combination will produce the fitting function sought. The data fitting is obtained by using function *linfit*. This function requires the following arguments: the vector of values of the independent variable Q , the vector of values of the dependent variable H , and the vector of functions $F(Q)$. The coefficients that accompany each function in $F(Q)$ are obtained by using

$$S := \text{linfit}(Q, H, F)$$

The function to be used for calculations is obtained by using $h(Q) := F(Q) \cdot S$. A plot of the original data and the fitted data is also shown to verify the fitting.



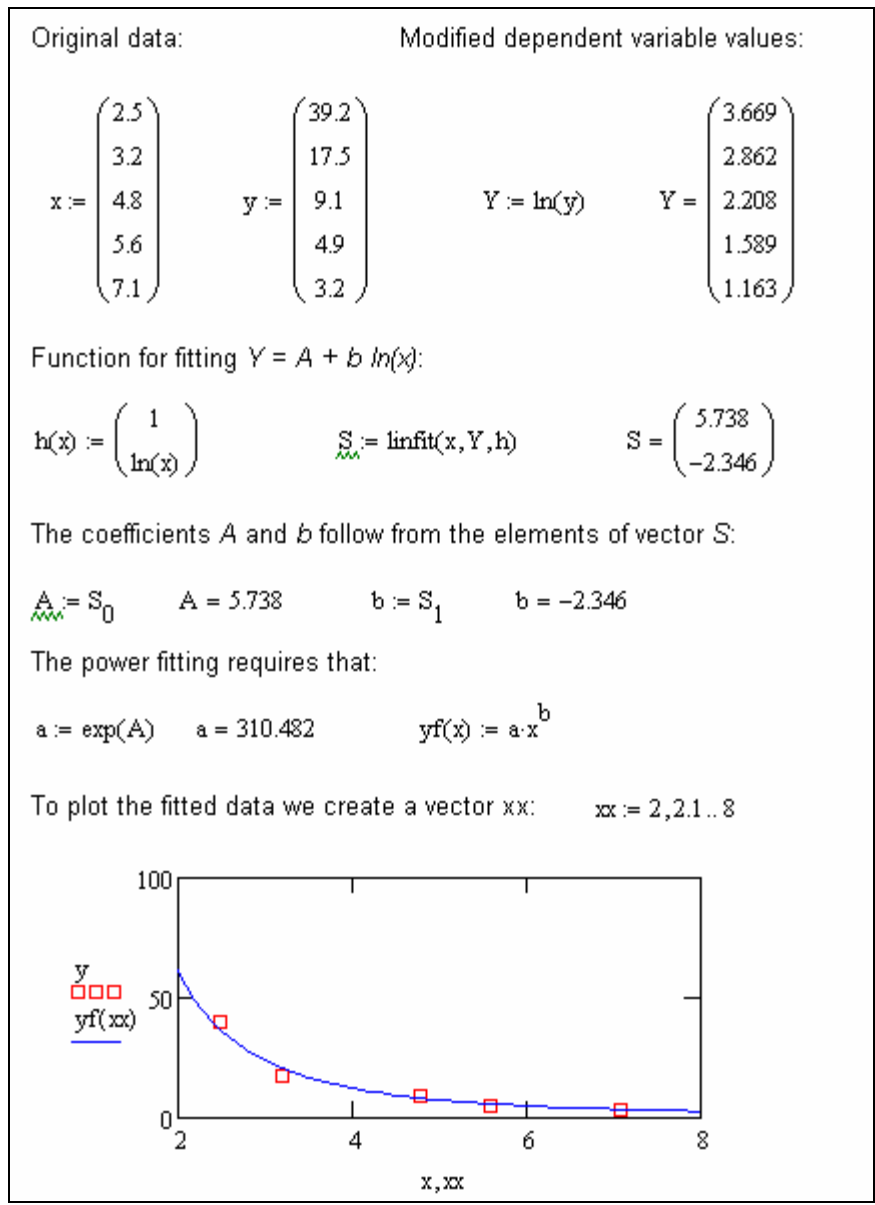
As a second example of application of *linfit* consider fitting x - y data to a power function of the form

$$y = ax^b.$$

Since *linfit* requires a linear combination of functions, we could take logarithms of the equation and write it as

$$\ln(y) = \ln(a) + b \ln(x).$$

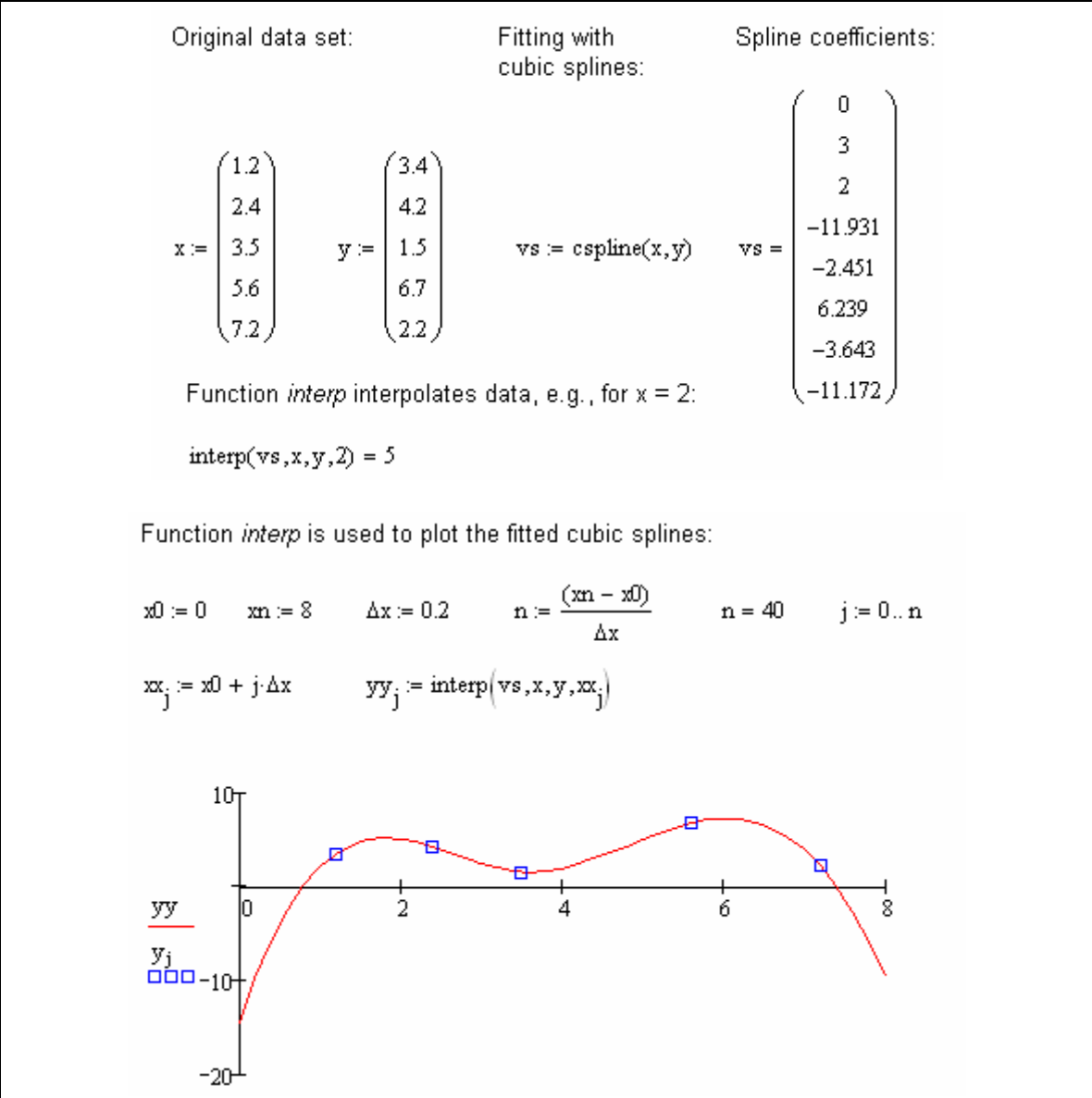
Thus, a linear fitting for this case would be the function $Y = A + b \ln(x)$, with $A = \ln(a)$ and $Y = \ln(y)$. The following *Mathcad* demonstrate the use of function *linfit* to recover a power function:



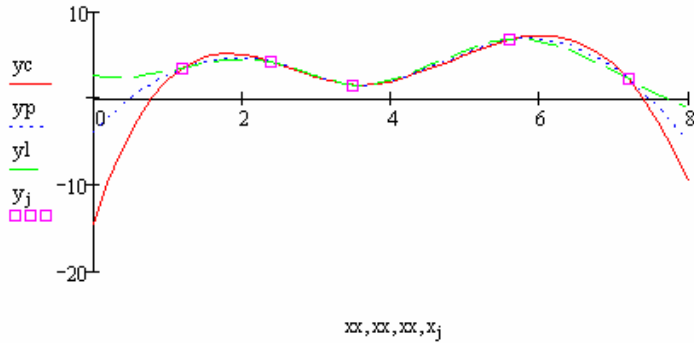
Curve interpolation with splines

Mathcad provides functions *cspline*, *pspline*, and *lspline* to perform cubic, parabolic, or linear spline interpolation. These functions require vectors of data x and y as arguments, and return a vector of coefficients for the splines. In the following example, we use cubic splines, and calculate the spline coefficient vector $vs := cspline(x,y)$.

Function *interp* is used to interpolate data out of the fitted splines. The call to *interp* is *interp(vs,x,y,xref)*, which returns the interpolated value of y at $x = xref$. The example below shows also how to use function *interp* to plot the fitted cubic splines:

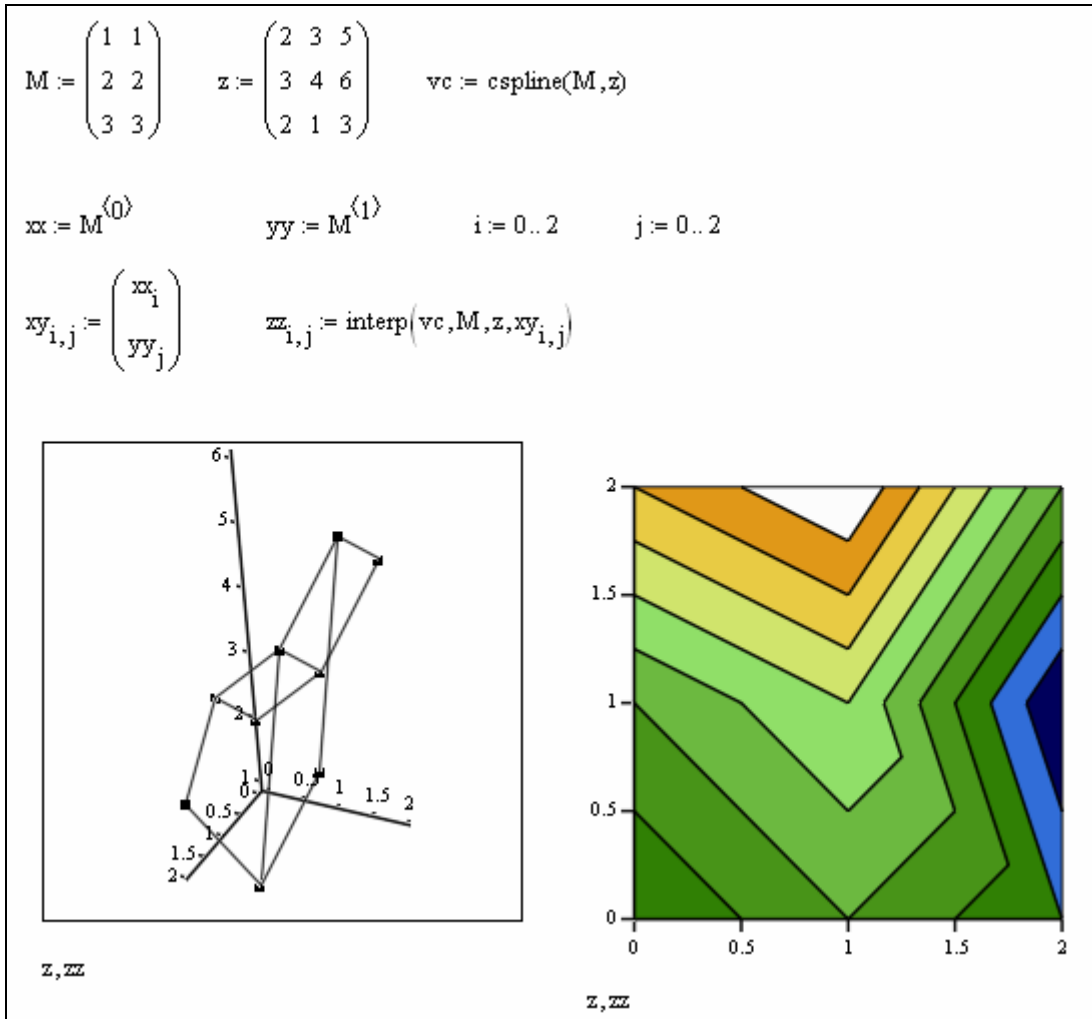


The reader is invited to repeat this exercise using the parabolic (*pspline*) and linear (*lspline*) spline fittings. The graph below shows all three spline fittings and the original data in the same set of axes:



Surface interpolation with splines

Functions *cspline*, *pspline*, and *lspline* can be used to interpolate spline surfaces by using, for example, the call *cspline*(*M*,*z*), where *M* is an *n*×*2* matrix describing the *x*-*y* grid, and *z* is an *n*×*n* matrix describing the surface values to be fitted. Columns 1 and 2 of matrix *M* represent the values of *x* and *y* defining the grid upon which the values of *z* are defined, i.e., $z_{i,j} = f(x_i, y_j)$. An example of spline surface interpolation is shown below:



Simple linear data fitting

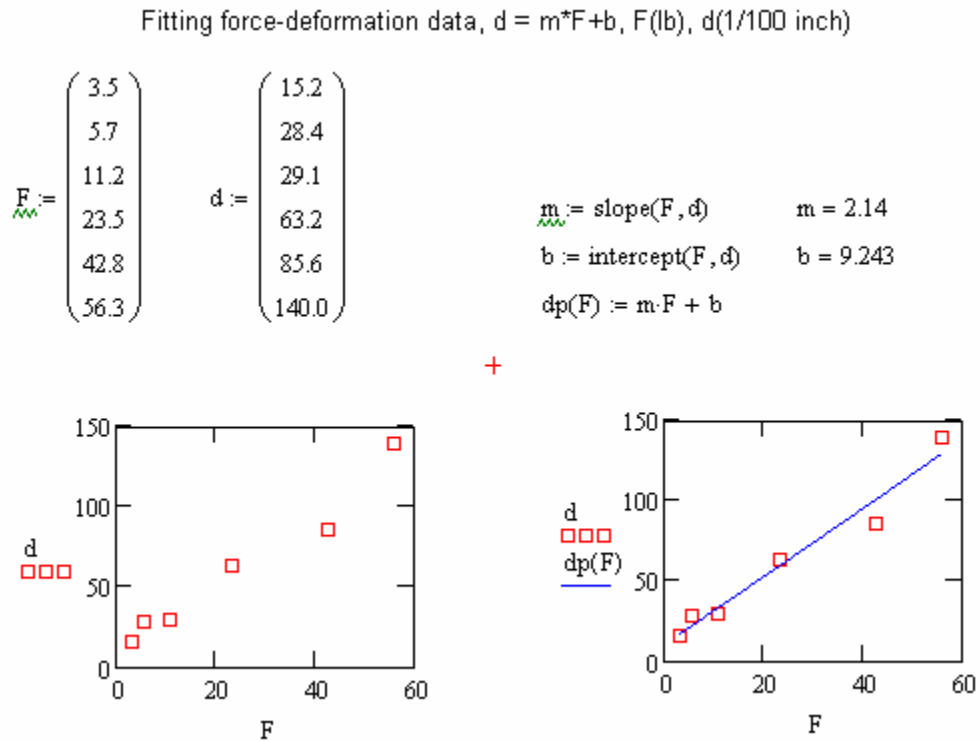
Given vectors of data *x* and *y* of the same length, a plot of *y*-vs-*x* could suggest a linear relationship of the form $y = mx + b$. In order to calculate the slope (*m*) and the intercept (*b*) for the linear data fitting, *Mathcad* provides functions *slope* and *intercept*, i.e.,

$$m := \text{slope}(x, y), \quad b := \text{intercept}(x, y)$$

A function $yp(x) := m * x + b$ can be defined to produce the data fitting.

The following example shows the calculation and graph of a linear data fitting for force-deformation data. First, the data is entered and a plot of the data is produced to check for

a linear relationship. Subsequently, *Mathcad* functions *slope* and *intercept* are used to calculate the parameters of the linear fitting. Finally, a plot of the original and fitted data is produced.



As a second example of application of functions *slope* and *intercept* consider fitting x - y data to an exponential function of the form

$$y = ae^{bx}.$$

This equation can be linearized by taking logarithms of both sides, i.e.,

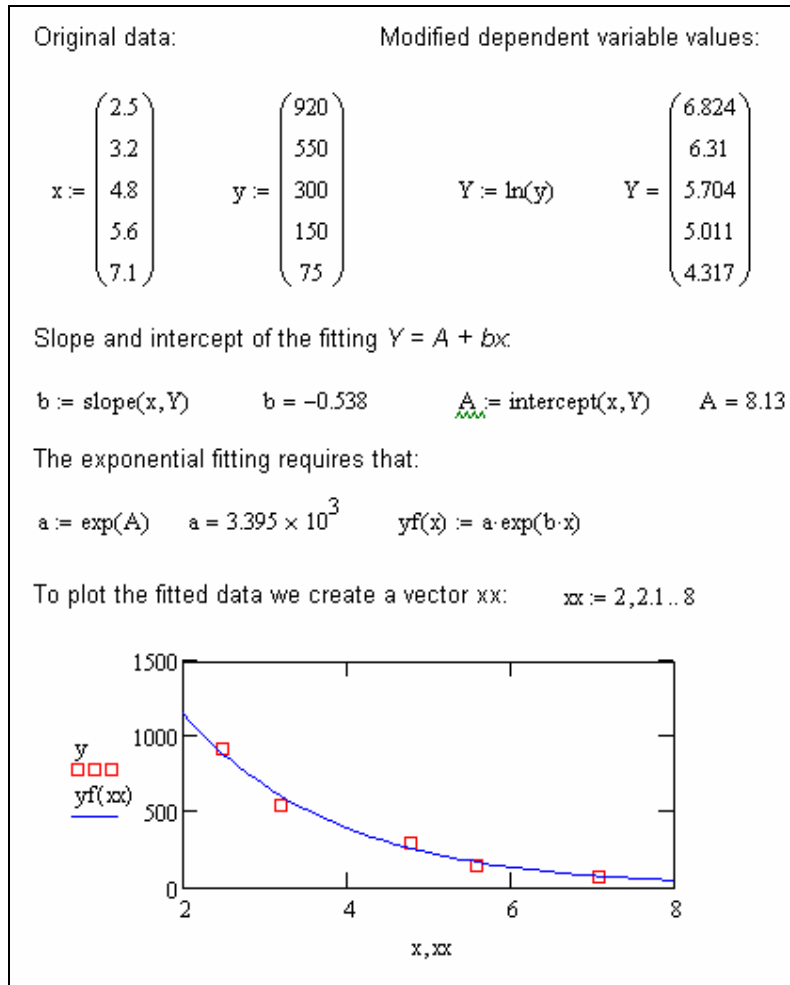
$$\ln(y) = \ln(a) + bx$$

Thus, a linear fitting for this case would be the function $Y = A + bx$, with $A = \ln(a)$ and $Y = \ln(y)$. Thus, for this case,

$$b := \text{slope}(x, \ln(y))$$

$$A := \text{intercept}(x, \ln(y)).$$

An example of fitting an exponential distribution is shown next:



Solution of first-order ordinary differential equations

A first-order ordinary differential equation (ODE) is an expression of the form

$$y'(t) = f(t, y(t)),$$

subject to the initial condition $y(t_0) = y_0$. An ODE of this type can be solved with function *Odesolve*, within a *Given* block, as illustrated in the following example. Notes: To enter the prime symbol in this case, use *Cntl F7*. Also, use *Cntl =* for the Boolean equal sign:

Given data: $f(t, y) := \frac{1}{y^2} + \sin(t) \quad t0 := 0 \quad y0 := 1$

End solution at: $tf := 25$ Number of points in solution: $n := 1000$

Given

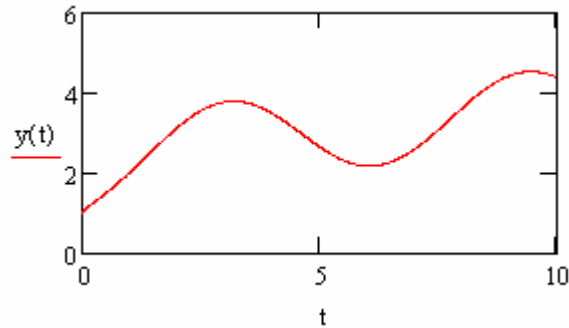
$$y'(t) = f(t, y(t)) \quad y(t0) = y0$$

$$y := \text{Odesolve}(t, tf)$$

The solution is available for evaluation, e.g.,

$$y(1) = 1.975 \quad y(2) = 3.094$$

or, it can be plotted as shown next:



Other functions for the solution of ODEs are *rkfixed*, *rkadapt*, and *radau*. The first two are *Runge-Kutta* solutions, one with fixed steps and one with adaptive steps, and the last one is used for stiff ODE systems.

Next, we illustrate the use of function *rkfixed* for numerical solution of a first-order ODE. Function *rkfixed* is called using

$$S:=rkfixed(Y0,t0,tf,n,D)$$

where $Y0$ is a vector of initial conditions, $t0$ and tf are the initial and ending values of the independent variable, n is the number of steps in the solution, and $D(t, Y)$ is the derivative function given in terms of the independent variable t and a number of unknown derivatives Y . The equation is set up as $dY/dt = D(t, Y)$, subject to $Y(t0) = Y0$. This description assumes that the solution Y is a vector of values $y(t)$, $y'(t)$, $y''(t)$, etc. This set up corresponds to the solution of an ODE of any order that has been transformed to a system of first-order ODEs. This approach will be illustrated below. Here, we show how to set up and solve the problem solved above with *Odesolve*, but now solved with *rkfixed*.

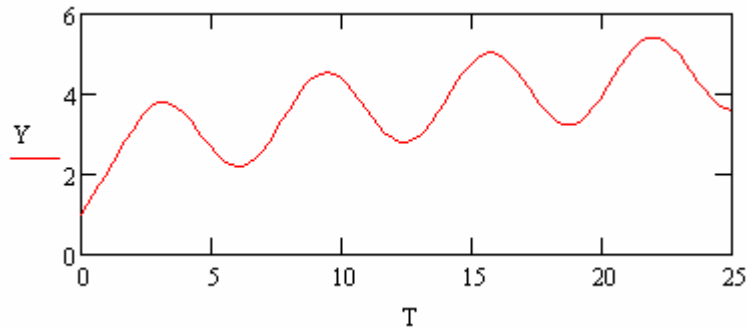
The following results show the setting up of the solution, and the use of function *rkfixed* to produce the solution matrix S . The values of t get stored into $T =$ first column (0) of S , while the solution y is contained in the second column (1) of S .

$$D(t,y) := \left(\frac{1}{y^2} \right) + \sin(t) \quad y_0 := 1 \quad t_0 := 0 \quad t_f := 25 \quad n := 100$$

$$S := \text{rkfixed}(y_0, t_0, t_f, n, D)$$

$$T := S^{(0)} \quad Y := S^{(1)}$$

A plot of the solution is shown next:



The following table, generated using transposes of matrix S , shows the first solution steps. The remaining of the solution can be seen by clicking on the table and scrolling to the right.

$$S^T =$$

	0	1	2	3	4	5	6	7	8	9
0	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25
1	1	1.233	1.463	1.709	1.975	2.256	2.544	2.828	3.094	3.331

Solving a second-order ODE with *rkfixed*

A second-order ODE of the form $y''(t) + a y'(t) + b y(t) = g(t)$, subject to the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, can be re-written in terms of a vector of unknowns U defined as follows: Let $u_1(t) = y(t)$, and $u_2(t) = y'(t)$, then

$$u_1'(t) = y'(t) = u_2(t),$$

$$u_2'(t) = y''(t) = g(t) - a u_2(t) - b u_1(t).$$

In vector-matrix form, this is written as

$$\frac{d}{dt} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \cdot \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

We can now define the following vectors and matrices:

$$U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \quad G(t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix},$$

With the ODE system to be solved now being

$$U'(t) = AU(t) + G(t) = D(t,U(t))$$

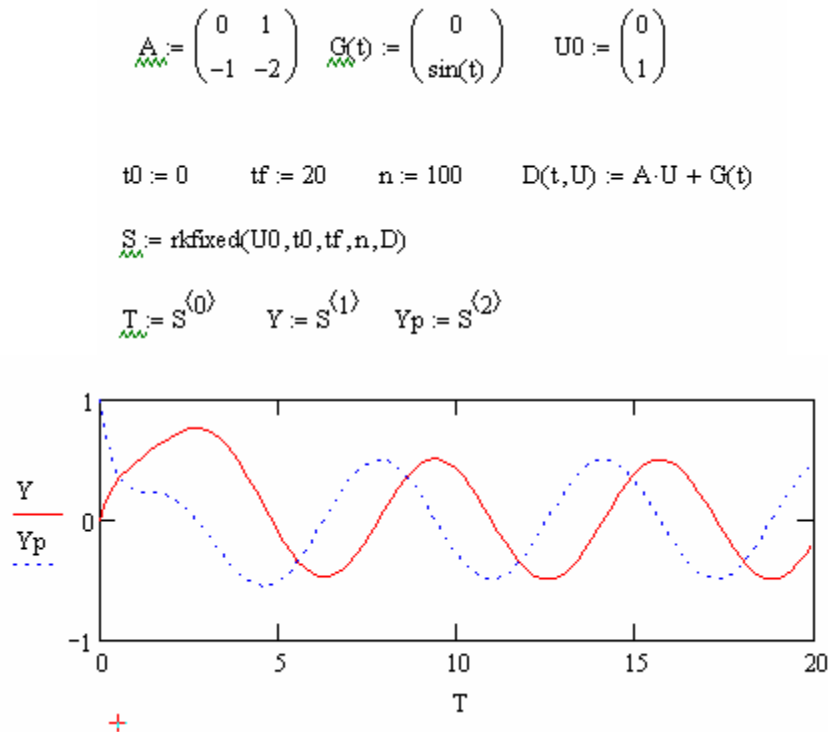
Subject to $U(t_0) = U_0 = [y_0, y_0']^T$.

This system can be solved with *rkfixed* as $S := rkfixed(U_0, t_0, t_f, n, D)$.

Consider the following case in which $a = 2$, $b = 1$, $t_0 = 0$, $y_0 = 0$, $y_0' = 1$, and $g(t) = \sin(t)$. The following results show the solution of the differential equation

$$y''(t) + 2y'(t) + y(t) = \sin(t), \quad y(0) = 0, \quad y'(0) = 1.$$

The first column of S represents the independent variable t , while the second and third columns represent $y(t)$ and $y'(t)$, respectively. The two dependent variables are shown in the graph.



This approach to the solution of a second-order ODE can be generalized to the solution of ODEs of any order, as long as it is linear.

ODE solution: gradually-varied flow (GVF) solution in a trapezoidal channel

The solution of gradually-varied flow (GVF) in open channels involves solving a first-order ODE after calculating the normal and critical flow depths. An example of GVF solution in a trapezoidal channel is described next. The figure describing the cross-section was created in *Paint* and added with the menu option *Insert>Picture*.

Calculation of Gradually-Varied Flow (GVF) in a trapezoidal channel.

Variables:

- b: bed width
- m_H : side slope
- y: depth
- S_o : bed slope

K: coefficient for Manning's equations

[K = 1.0 (SI), K = 1.489 (ES)]

n: Manning's n

Q: discharge

g: acceleration of gravity

[g = 9.81 (SI), g = 32.2 (ES)]

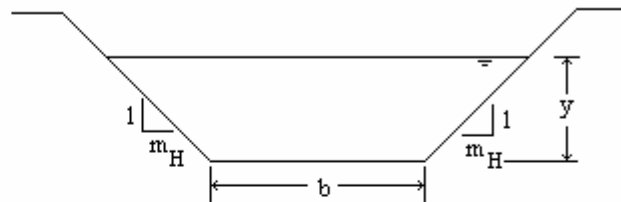
y_0 : initial depth for calculation, i.e.,

depth at section x_{start} .

MAKE SURE YOU USE THE PROPER UNIT SYSTEM. Give the following

data:

b := 15.2 $m_H := 0$ $S_o := 0.0009$ $K_{ww} := 1.0$ n := 0.022 Q := 107 $g_{ww} := 9.81$



- x_{start} : initial position for calculation
- x_{end} : ending position for calculation
- Δx : uniform step size
- N: number of cross-sections in calculation
- z_0 : bed elevation at section x_{start}

+

First, determine the type of uniform flow to be expected in this channel.

Normal depth (y_n) calculation:

Assume $y_n := 1.0$

Given

$$Q = \frac{[(b + m_H y_n) \cdot y_n]^{\frac{5}{3}} \sqrt{S_o} \cdot \frac{K}{n}}{(b + 2 \cdot y_n \cdot \sqrt{1 + m_H^2})^{\frac{3}{2}}}$$

$y_n := \text{Find}(y_n)$ $y_n = 3.066$

Critical depth (y_c) calculation:

Assume $y_c := 1.0$

Given

$$\frac{[(b + m_H y_c) \cdot y_c]^3}{b + 2 \cdot m_H y_c} = \frac{Q^2}{g}$$

$y_c := \text{Find}(y_c)$ $y_c = 1.716$

G.V.F. calculation follows. Determine the type of flow and backwater curve that is appropriate according to the relative values of y_n , y_c and y_0 , and decide direction of calculation (upstream or downstream). Fill in the following information:

* The slope is subcritical, calculation will proceed upstream with a curve type M3.

Give
these
data:

$$x_{\text{start}} := 100.0$$

$$x_{\text{end}} := 0.0$$

$$N := 100$$

$$\Delta x := \frac{x_{\text{end}} - x_{\text{start}}}{N}$$

$$y_0 := 0.98$$

$$z_0 := 0.0$$

$$\Delta x = -1$$

Solution to the differential equation using the *rkfixed* (Runge-Kutta) function in Mathcad:

$$D(x,y) := \frac{S_0 - \left(\frac{Q \cdot n}{K}\right)^2 \frac{\left(b + 2 \cdot y_0 \cdot \sqrt{1 + m_H^2}\right)^{\frac{4}{3}}}{\left[(b + m_H y_0) \cdot y_0\right]^{\frac{10}{3}}}}{1 - \left(\frac{Q^2}{g}\right) \frac{b + 2 \cdot m_H y_0}{\left[(b + m_H y_0) \cdot y_0\right]^3}}$$

The function $D(x,y)$ defines the differential equation to be solved:
 $dy/dx = (S_f - S_0)/(1 - Fr^2)$

$$Z := \text{rkfixed}(y, x_{\text{start}}, x_{\text{end}}, N, D)$$

Call to function *rkfixed* to solve differential equation using a Runge-Kutta method.

$$Z_{0,2} := z_0$$

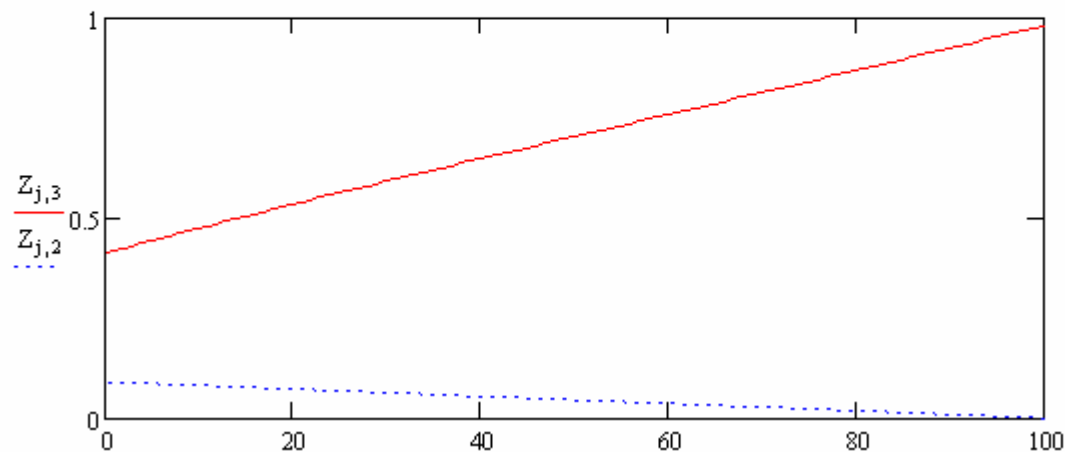
$$j := 0..N$$

$$Z_{j,2} := Z_{0,2} - j \cdot \Delta x \cdot S_0 \quad \leftarrow \text{bed elevation}$$

Calculate bed and water surface elevations for the channel reach used in the GVF calculation.

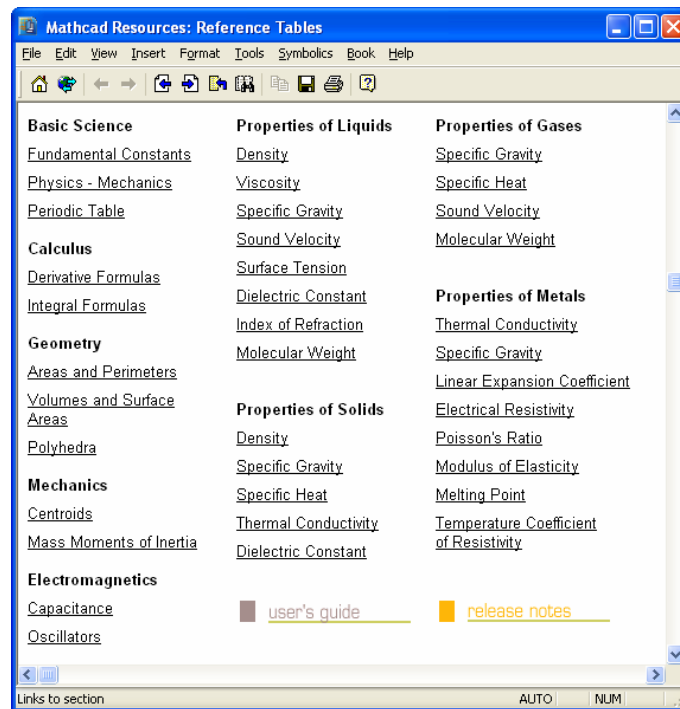
$$Z_{j,3} := Z_{j,1} + Z_{j,2} \quad \leftarrow \text{W.S. elevation}$$

Plot channel bed and free surface profile (not to

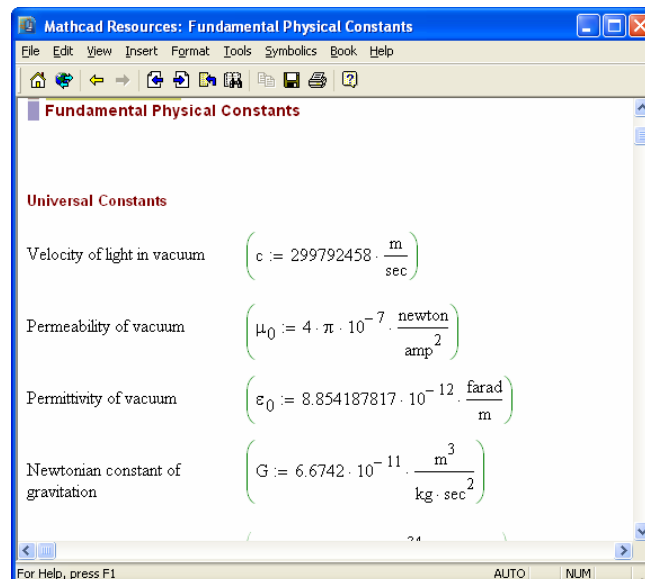


Mathcad reference tables and User's Guide

Mathcad provides a nice collection of reference tables by choosing the *Help>Reference Tables*. The options available are shown below:



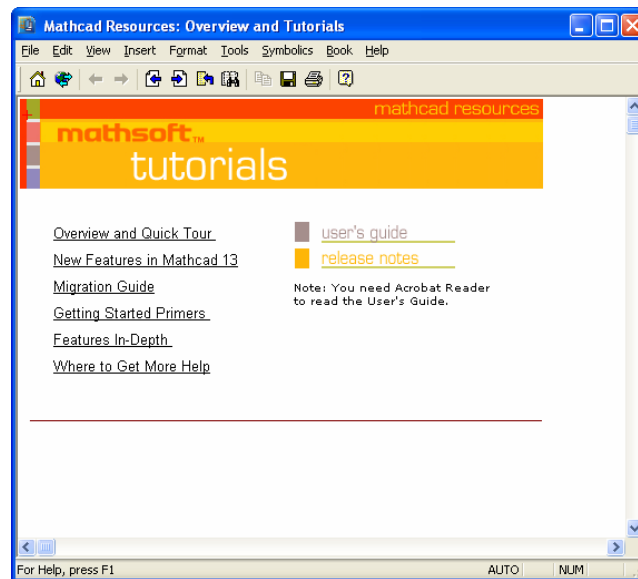
Clicking the *Fundamental Constants* link, for example, produces the following list of constants:



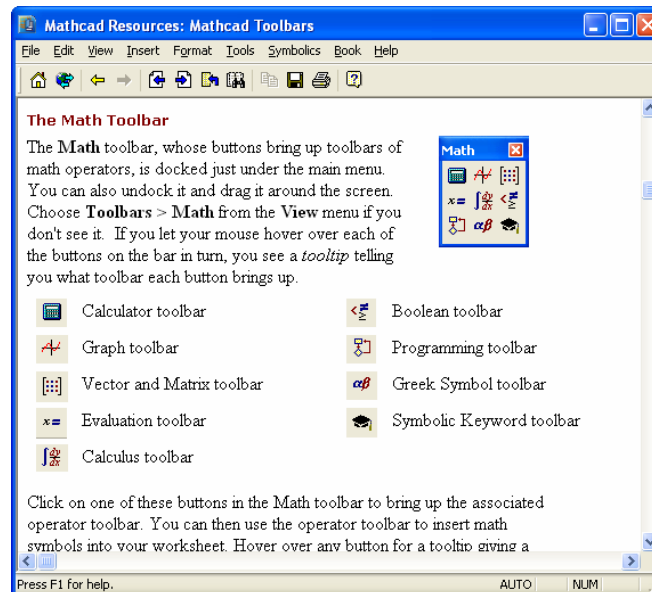
Explore the reference tables on your own to identify those that may be helpful to your current classes or applications. Scrolling down the *Reference Tables* one finds the *user's guide* link which provides detailed instructions on the use of *Mathcad's* different features.

Using Mathcad's Tutorials

Tutorials are Mathcad worksheets that provide basic instructions on the use of Mathcad functions. Select the *Help>Tutorials* to access the available worksheets:



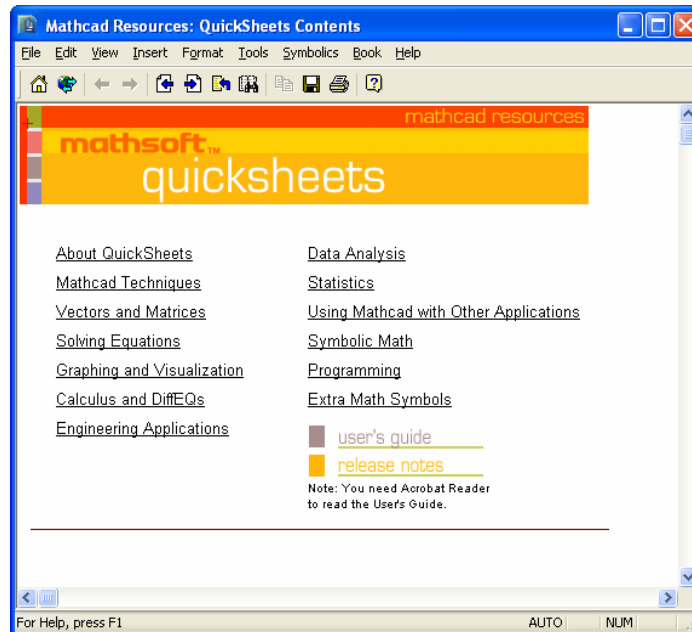
A good place to explore for new users would be the *Getting Started Primers* link. This will open a menu of links including one entitled *Mathcad Toolbars*. Clicking on this link provides a worksheet describing the different toolbars. For example, scrolling down the worksheet provides the following description of the *Math Toolbar* (some of which buttons we have presented earlier):



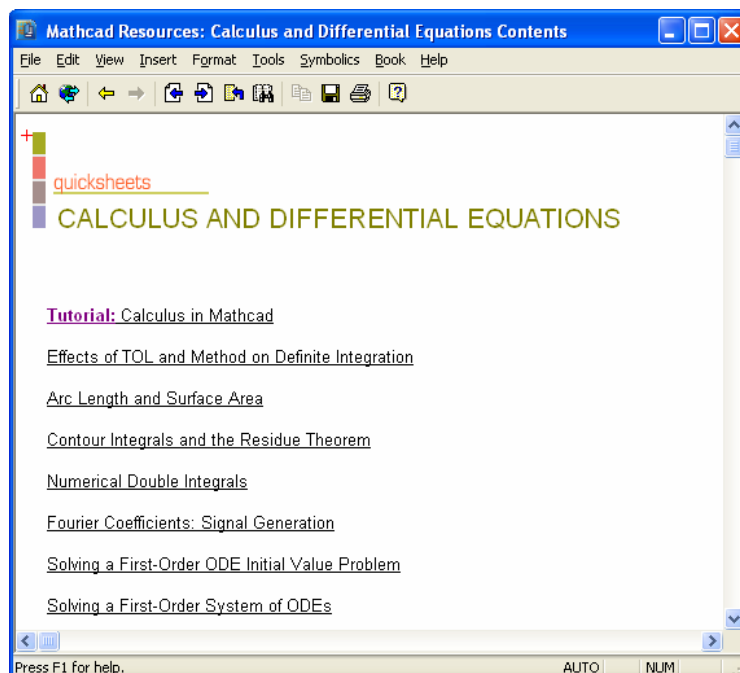
Explore more of the *Getting Started Primers* on your own. The menu in this *Tutorial* worksheet includes some of the subjects addressed in this document, e.g., *Defining Functions*, etc.

Using Mathcad's QuickSheets

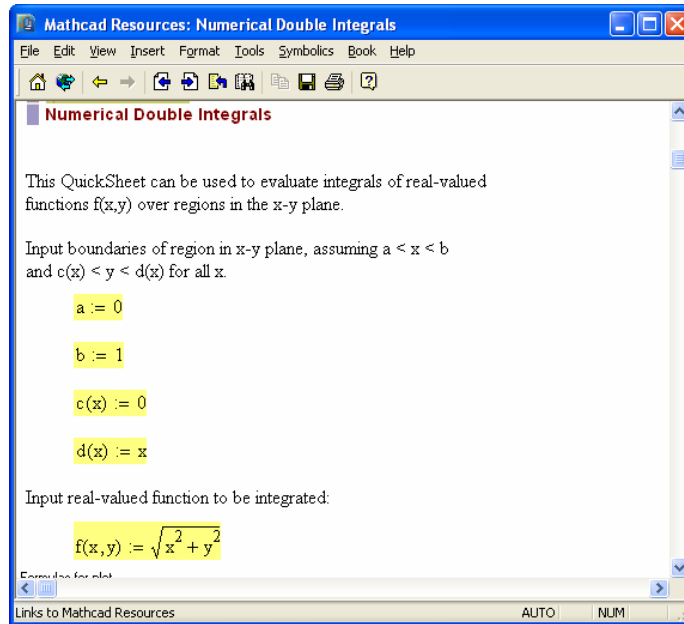
QuickSheets are reference *Mathcad* worksheets that contain solved problems on a given mathematical subject. To access *QuickSheets* select the menu option *Help>QuickSheets*. This action will open a menu of *QuickSheets* as shown below:



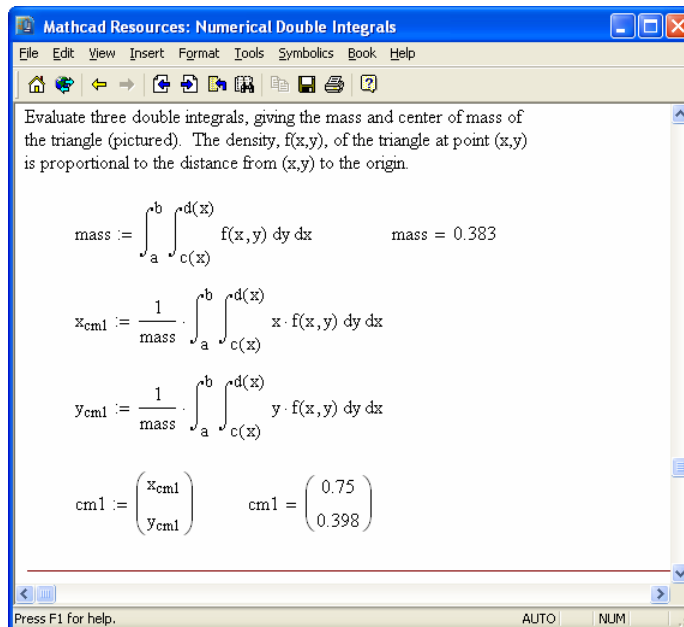
The *QuickSheets* menu list a number of subjects for the user to select. For example, if you are looking for an example for calculating a double integral, you could click on the *Calculus and DiffEQs* link to access the following menu:



Select the *Numerical Double Integrals* link to see an example. The example, whose set up is shown below, describes triangular-shaped plate described by $R = \{c(x) < y < d(x), a < x < b\}$ with a surface mass density $f(x,y)$.



Scrolling down the *QuickSheet*, we find a plot of the integration region as well as the following calculations for the mass and center-of-mass coordinates for a triangular plate with are density $f(x,y)$, i.e.,



Using the *File>Save As...* menu option you can copy the *QuickSheet* into a new *Mathcad* worksheet that you can then modify to address a specific solution.